UNIVERSE+ School
Exploring Positive Geometry

Claudia Fevola: Euler integrals 101
Lecture 1

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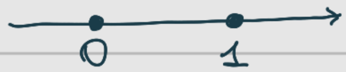
EULER INTEGRALS 101

Q: What is this useful for?
If I like algebra, geometry, combinatorics why should I study integrals?

EXAMPLE 0: Euler beta function



$$B(\nu, 1-s) = \int_0^1 \frac{x^{\nu}}{(1-x)^{s}} \frac{dx}{x} = \frac{\Gamma(\nu)\Gamma(1-s)}{\Gamma(\nu+1-s)}$$

where $\Gamma(u) = \int_0^{\infty} t^{u-1} e^{-t}\, dt$ gamma function $\qquad X = (\mathbb{C}^*) \setminus \{1\}$

Ex 1: $\mathcal{M}_{0,5}$ is the moduli space of 5 marked points on $\mathbb{P}^1$ (up to automorph. $PSL(2)$)

A parametrization of $\mathcal{M}_{0,5}$ is given by

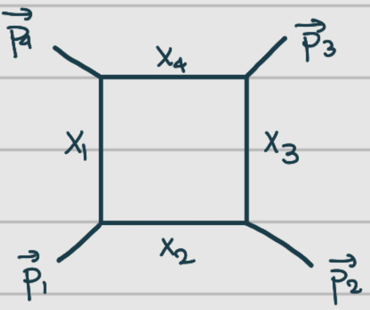
$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1+x_1 & 1+x_1+x_2 & 1 \end{pmatrix}$$

$$f_{13} = 1+x_1, \quad f_{14} = 1+x_1+x_2, \quad f_{23} = x_1, \quad f_{24}(x_1+x_2), \quad f_{34} = x_2$$

5-point string amplitude

$$\mathcal{I}_5 = (\alpha')^2 \cdot \int_{\mathbb{R}^2_+} \frac{x_1^{\alpha'\nu_1} x_2^{\alpha'\nu_2}}{(1+x_1)^{\alpha' s_{13}} (1+x_1+x_2)^{\alpha' s_{14}} (x_1+x_2)^{\alpha' s_{24}}} \frac{dx_1\, dx_2}{x_1 x_2}$$

Ex 2: Massless box diagram in Lee-Pomeransky representation



$$\mathcal{G} = \frac{1}{2}\begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & s & 0 \\ 1 & 0 & 0 & 0 & t \\ 1 & s & 0 & 0 & 0 \\ 1 & 0 & t & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \mathcal{U} + \mathcal{F}$$

$$s = (p_1+p_2)^2$$
$$t = (p_2+p_3)^2$$

$$\mathcal{I}_{box} = N_\nu \int_{\mathbb{R}^4_{>0}} \frac{x^{\nu-1}}{\mathcal{G}^{-D/2}}\, dx,$$

$$dx = dx_1 \dots dx_4$$
$$x^{\nu-1} = x_1^{\nu_1-1} x_2^{\nu_2-1} \dots x_4^{\nu_4-1}$$

$$X_{s,t} = (\mathbb{C}^*)^4 \setminus \{\mathcal{G}(s,t;x)=0\}$$

Ex 3: Cosmological correlators

An Euler integral is an integral of the form

$$\int_\Gamma \frac{x_1^{\nu_1} \cdots x_n^{\nu_n}}{f_1^{s_1} \cdots f_\ell^{s_\ell}} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} = \int_\Gamma f^{-s} x^\nu \frac{dx}{x}$$

$f_1, \dots, f_\ell \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ Laurent polynomials

$$(s,\nu) = (s_1, \dots, s_\ell, \nu_1, \dots, \nu_n) \in \mathbb{C}^{\ell+n}$$

Ordinary theory of single-valued functions is formalized under the name of de Rham theory. By modifying this theory we will construct a theory suitable for multivalued

Question: How to formulate Stokes' theorem for integrals of multivalued functions?

diff form of one deg lower

$$\boxed{\int_{\Delta} d\psi = \int_{\partial\Delta} \psi} = 0$$

if ψ vanishes on boundary

smooth chain

$X = \mathbb{C}^n \setminus D$ $\qquad D_j = \{f_j(x) = 0\}$

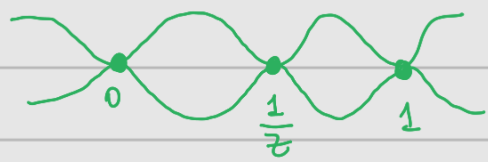
$D = \bigcup_{j=1}^{m} D_j$ $\qquad f_j(x) = f_j(x_1, \dots, x_n) \quad 1 \leq j \leq \ell$

Consider a multivalued function on X

$$f^{-s}(x) = \prod_{j=1}^{m} f_j(x)^{-s_j}, \qquad s_j \in \mathbb{C} \setminus \mathbb{Z}, \quad 1 \leq j \leq \ell$$

1^{st} idea (bad idea): lifting f to a covering manifold $\tilde{X}$ of X so that f becomes single valued.

is there a better pict? maybe Riemann surface?



Ex: $s_j = -\frac{1}{2}$,

$$\Rightarrow \tilde{X} = V(y^2 = x(1-x)(1-zx))$$

Relation between X and $\tilde{X}$ can be very hard.

2^{nd} idea: Consider $\omega = d\log(f^{-s}) =$

$$= \sum_{j=1}^{\ell} d\log(f_j^{-s_j}) = \sum_{j=1}^{m} -s_j \, d\log(f_j) = \sum_{j=1}^{m} -s_j \frac{df_j}{f_j}$$

Ex 0 $\quad d\log(x^{\nu}(1-x)^{-s}) = \nu \frac{dx}{x} + s \frac{dx}{(1-x)}$

single valued holomorphic 1-form on M

INTUITIVE EXPLAINATION

Δ singular k-simplex on X

φ smooth k-form on X

$\varphi \in \Omega^k(X)$

To determine the integral of a multivalued k-form $f\varphi$ we need to fix a branch of f on Δ

$\Delta \otimes f_\Delta$ $\rightarrow$ symbol to say we are fixing a branch of f on Δ

$$\int_{\Delta \otimes U_\Delta} f \cdot \varphi := \int_\Delta [\text{the fixed branch } f_\Delta \text{ of } f \text{ on } \Delta] \cdot \varphi$$

Since f_Δ can be continued analytically on a sufficiently small neighborhood of Δ, on this neighborhood, for a single valued p-form $U_\Delta \cdot \varphi$ and Δ, ordinary Stokes hold:

For $\varphi \in \Omega^{k-1}(X)$

$$\int_{\Delta \otimes f_\Delta} f \cdot \nabla_\omega \varphi = \int_\Delta d(f_\Delta \cdot \varphi) = \int_{\partial \Delta} f_\Delta \cdot \varphi$$

$$d(f_\Delta \cdot \varphi) = df_\Delta \wedge \varphi + f_\Delta \, d\varphi = f_\Delta \overbrace{\left(d\varphi + \frac{df_\Delta}{f_\Delta} \wedge \varphi\right)}^{\nabla_\omega \varphi = (d + \omega \wedge)\varphi} = f_\Delta \nabla_\omega \varphi$$

on Δ.

$\omega = \frac{df}{f}$ single valued holomorphic 1-form

Now rewrite above with new symbol

Ex: $\nabla_\omega \cdot \nabla_\omega = 0$

Then Stokes becomes

$$\boxed{\int_{\Delta \otimes U_\Delta} f \cdot \nabla_\omega \varphi = \int_{\partial_\omega(\Delta \otimes f_\Delta)} f \cdot \varphi}$$

Note: the underlying forms are the same but the rules for exactness have been changed.

However in our setting f vanishes on the boundary so

$$\int_{\Gamma = \Delta \otimes U_\Delta} U \cdot \nabla_\omega \varphi = 0$$

smooth k-form on M

How do k-forms on X look like?

$$\Omega^k(X) = \left\{ \sum_{i_1 < \dots < i_k} h_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k} \;\middle|\; h_{i_1, \dots, i_k} \in \underbrace{\sum_{\substack{a \in \mathbb{Z}^\ell \\ b \in \mathbb{Z}^n}} \mathbb{C} \cdot f^a x^b}_{\Omega^0(X)} \right\}$$

regular k-forms on X

EXAMPLE: $\omega = d\log(f^{-s} x^\nu) \in \Omega^1(X)$

Ex 0 : $B(a+\nu, 1-s-b) = \int_0^1 \frac{x^{\nu+a}}{(1-x)^{s+b}} \frac{dx}{x}$ $\quad a,b \in \mathbb{Z}$

$$f(x) = x^{\nu+a-1}(1-x)^{-s-b}$$

$$0 = \int_0^1 f \cdot \nabla_\omega(1) = \int_0^1 \frac{x^{\nu+a-1}}{(1-x)^{s+b}} \cdot \left(\frac{s}{1-x} + \frac{\nu}{x}\right) dx =$$

$$= s \int_0^1 \frac{x^{\nu+a}}{(1-x)^{s+b+1}} \frac{dx}{x} + \nu \cdot \int_0^1 \frac{x^{\nu+a-1}}{(1-x)^{s+b}} \frac{dx}{x}$$

$$\Longrightarrow \quad B(a+\nu, -s-b) = -\frac{\nu}{s} \cdot B(a+\nu-1, -s-b+1) \quad \square$$

In general, for $\phi \in \Omega^{n-1}(X)$

$$\int_\Gamma x^{\nu-1} f^{-s} \nabla_\omega(\phi) = 0$$

where $\nabla_\omega(\phi) = \sum_{(a,b)} c_{a,b}(\phi)\, x^a f^b \, dx_1 \wedge \ldots \wedge dx_n$

with finitely many non-zero coefficients

This identity gives a $\mathbb{C}$-linear relation

$$\sum_{(a,b)} c_{a,b}(\phi)\, I_{a,b} = 0$$

IBP RELATIONS

So we have seen that to each $\phi \in \Omega^{n-1}(X)$ corresponds a relation

Hence, the IBP relations correspond to $\mathrm{im}(\nabla_{n-1})$

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