UNIVERSE+ School Exploring Positive Geometry

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Lecture 2

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LECTURE 2:

$$\mathcal{I}_{a,b} := \int_\Gamma f^{-s+a} x^{\nu+b} \frac{dx}{x} = \int \prod_{i=1}^{\ell} f_i(x)^{-s_i+a_i} \cdot x_1^{\nu_1+b_1} \cdots x_n^{\nu_n+b_n} \frac{dx}{x}$$

$$X = (\mathbb{C}^*)^n \setminus \{f_1 = \dots = f_\ell = 0\}$$

$$= \{x \in \mathbb{C}^n : x_1 \cdots x_n \cdot f_1(x) \cdots f_\ell(x) \neq 0\} \qquad (*)$$

$$V_\Gamma := \mathrm{Span}_{\mathbb{C}} \left\{ [\Gamma] \longmapsto \mathcal{I}_{a,b} \right\}_{(a,b) \in \mathbb{Z}^\ell \times \mathbb{Z}^n}$$

$$\phi \in \Omega^{n-1}(X) \xrightarrow[\nabla_\omega(\phi)]{\text{Stokes}} \sum_{(a,b)} c_{a,b}(\phi)\, \mathcal{I}_{a,b} = 0$$

IBP RELATIONS

As $\nabla_\omega \cdot \nabla_\omega = 0$.

Def: The (algebraic) twisted de Rham complex is

$$(\Omega^\bullet(X), \nabla_\omega) : 0 \xrightarrow{\nabla_\omega} \Omega^0(X) \longrightarrow \dots \xrightarrow{\nabla_\omega} \Omega^{n-1}(X) \xrightarrow{\nabla_\omega} \Omega^n(X) \to 0$$

$$H^k(X,\omega) = \frac{\{\phi \in \Omega^k(X) : \nabla_\omega(\phi) = 0\}}{\nabla_\omega \Omega^{k-1}(X)} = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}$$

In particular $H^n(X,\omega) = \dfrac{\Omega^n(X)}{\nabla_\omega(\Omega^{n-1}(X))}$

Analytic: complex manifold holomorphic differential forms.
CANONICALLY ISOMORPHIC

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} H^k(X,\omega) \underset{\uparrow}{=} (-1)^n \dim H^n(X,\omega)$$

Vanishing theorem

$$\dim H^k(X,\omega) = 0, \ \forall k \neq n$$

Theorem: Let X be as in $*$. Fix generic $(s,\nu) \in \mathbb{C}^{\ell+n}$ and $\omega = d\log(f^s x^\nu)$. We have

$$\dim V_\Gamma = \dim_{\mathbb{C}} H^n(X,\omega) = |\chi(X)|$$

HOW DO WE COMPUTE THIS?

Def: A very affine variety is a closed subvariety of an algebraic torus $(\mathbb{C}^*)^N$

Ex:
1. $(\mathbb{C}^*)^n$
2. $\mathbb{C} \setminus \{0,1\}$, i.e. $\{x \in \mathbb{C} : x \neq 0,1\}$

$$\begin{aligned} \mathbb{C}\setminus\{0,1\} &\longrightarrow (\mathbb{C}^*)^2 \\ x &\longmapsto (x, x-1) \end{aligned} \qquad U = \{(x,y) \in (\mathbb{C}^*)^2 : y = x-1\}$$

3. complements of hyperplanes

$$\mathcal{A} = (\mathbb{C}^*)^n \setminus \bigcup_{i=1}^{\ell} H_i \ , \quad H_i = \{L_i(x) = 0\}$$

L_i have degree 1

4. Exercise: Think of how to embedd X in a torus.

5. $Gr(2,n)$ space of 2-diml linear subspaces of $\mathbb{C}^n$

$$Gr(2,n) = \{A \in Mat_{2\times n} : rk(A) = 2\} / GL_2$$

$$Gr(2,n)^0 = \{L \in Gr(2,n) : p_{ij}(L) \neq 0 \ \forall i<j\}$$

2x2 minors

"algebraic variety on which some regular functions are forced to stay $\neq 0$.

Indeed the natural setting for our integrals: $f_i^{\alpha_i}, d\log(f_i)$

$\boxed{\begin{array}{l}\text{inclusion-exclusion}\\ \chi(M\cup N)=\chi(M)+\chi(N)-\chi(M\cap N)\\ \text{fibration property}\\ \phi: E\to B \quad \chi(E)=\chi(B)\cdot\chi(F)\end{array}}$

<u>Ex</u>: 1. $\chi(\mathbb{C}) = \chi(pt) = 1$

$\mathbb{C}\simeq\mathbb{R}^2$
contractible to a point

2. $\chi(\mathbb{C}^*) = \chi(\mathbb{C}) - \chi(\{0\}) = 0$

3. $\chi(\mathbb{C}\setminus\{0,1\}) = 1-2 = -1$

etc: $\chi((\mathbb{C}^*)\setminus\{a_1,\dots,a_m\}) = m-1$

4. $\mathcal{M}_{0,5} \longrightarrow \mathcal{M}_{0,4}$

$[p_1,\dots,p_5] \longmapsto [p_1,\dots,p_4]$

$\chi(\mathcal{M}_{0,5}) = 2$

$$\begin{aligned}\chi(\mathcal{M}_{0,5}) &= \chi(\mathcal{M}_{0,4})\cdot\chi(F) = \\ &= \chi(\mathcal{M}_{0,4})\cdot\chi(\mathbb{P}^1\setminus\{4\text{ pts}\}) \\ &= \chi(\mathcal{M}_{0,4})\cdot(2-4) = \\ &= \chi(\mathcal{M}_{0,4})\cdot(-2)\end{aligned}$$

<u>Ex</u>: $\chi(\mathcal{M}_{0,m})$?

finding model parameters that best explain a given sequence of observations (done by maximizing L)

Parenthesis : MAXIMUM LIKELIHOOD ESTIMATION IN ALGEBRAIC STATISTICS

$\mathbb{P}^{n-1}[p_1:\dots:p_n]$ projective space with coordinates p_i

p_i represents the probability of the i-th event.

Def: An implicit statistical model is a closed subvariety $V \subseteq \mathbb{P}^{n-1}$

DATA : $(u_1, u_2, \dots, u_n) \in \mathbb{Z}_{\geq 0}$, $u_i = \#$ times the i-th event was observed.

GOAL : finding values of p_i on V which best explain data

STRATEGY : find the critical points of

$$L(p_1, \dots, p_n) = \frac{p_1^{u_1} \cdots p_n^{u_n}}{(p_1 + \dots + p_n)^{u_1 + \dots + u_n}}$$

likelihood function

$$\text{ML degree} = \left| \left\{ \begin{array}{l} \text{critical pts of } L|_V \text{ which are} \\ \text{not zero or poles of } L \text{ (and } u_i \\ \text{sufficiently general)} \end{array} \right\} \right|$$

$$= \left| \left\{ \text{critical pts of } L \text{ on } U := \{ x \in V : p_1 \cdots p_n (p_1 + \dots + p_n) \neq 0 \} \right\} \right|$$

this comes by wanting to solve a constrained optimization pb of maximizing L subject to $V_{>0} = V \cap \Delta_n$

Our setting: $X = (\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell)$

$$L = f^s x^\nu = f_1^{s_1} \cdots f_\ell^{s_\ell} x_1^{\nu_1} \cdots x_n^{\nu_n}$$

holomorphic function on U

mldegree of X = # critical pts of L for sufficiently general s, ν.

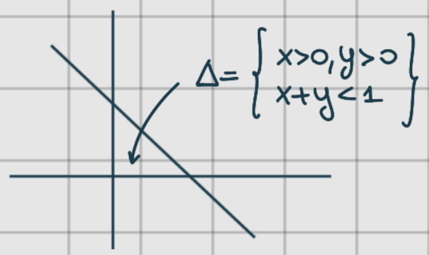
($\exists F \neq 0$ s.t. $F(u_1, \dots, u_n) \neq 0$)

Thm [Huh]: If X is a smooth very affine variety of dimension d, then the ML degree of X equals the signed Euler characteristic $(-1)^d \cdot \chi(U)$

"smooth is important: ex 1.4. in "Likelihood geometry"

Ex.: 1. $\mathcal{A} = \{x=0\} \cup \{y=0\} \cup \{x+y=1\}$

$$X = \mathbb{C}^2 \setminus \{xy(x+y-1)=0\}$$



$$L = \log\left(x^{\nu_1} y^{\nu_2} (x+y-1)^s\right) \qquad \nu_1, \nu_2, s > 0$$

$$\begin{cases} \dfrac{\nu_1}{x} + \dfrac{s}{x+y-1} = 0 \\ \dfrac{\nu_2}{y} + \dfrac{s}{x+y-1} = 0 \end{cases} \Longrightarrow \left(\frac{\nu_1}{\nu_1+\nu_2+s}, \frac{\nu_2}{\nu_1+\nu_2+s} \right) = (\hat{x}, \hat{y})$$

is a solution

$H_L(\hat{x}, \hat{y}) < 0 \Longrightarrow$ unique maximum on Δ

negative definite

$$\chi(U) = \chi(\mathbb{C}^2) - \chi(L_1 \cup L_2 \cup L_3)$$

$$= \chi(\mathbb{C}^2) - \chi(L_1) - \chi(L_2) - \chi(L_3) + \chi(p_{12}) + \chi(p_{13}) + \chi(p_{23})$$

$$= 1 - 3 + 3 = 1$$

Thm [Zalavsky]: The bounded regions in the complement of the arrangement in $\mathbb{R}^n$ are counted by $|\chi(X)|$

Thm [Varchenko]: The ML degree is the number of bounded regions

2. $\mathcal{M}_{0,n} = Gr(2,m)^0 / (\mathbb{C}^*)^m$

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & x_1 & x_2 & & x_{m-3} & 1 \end{bmatrix}$$

$$\mathcal{L}(x) = \sum_{(i,j)} s_{ij} \log(f_{ij}(x))$$ SCATTERING POTENTIAL

Ex: Try making up your favourite hyperpl. arrangement.

How can we compute these critical points?
Idea is to move from equations to ideals so we can use computer algebra

Recall: critical point equations

$$\frac{s_1 \frac{\partial f_1}{\partial x_i}}{f_1} + \dots + \frac{s_\ell \cdot \frac{\partial f_\ell}{\partial x_i}}{f_\ell} + \frac{v_i}{x_i} = 0 \quad , \quad i = 1, \dots, n$$

After clearing denominators we get a system of polynomial equations

$$F_1 = \dots = F_n = 0$$

However, we might have introduced spurious solutions on

$$D = \{ x_1 \cdots x_n f_1 \cdots f_\ell = 0 \}$$

In computer algebra, this is taken care by saturation

R ring, I ideal of R, $f \in R$

$$(I, f^\infty) = \{ g \in R : f^N g \in I \text{ for some } N \gg 0 \}$$

geometrically removes the components of $V(I)$ that are contained in $V(f)$

Ex: $I = (xy) \subset \mathbb{C}[x,y]$

$$V(I) = \{x = 0\} \cup \{y = 0\}$$

Suppose we want to keep only the part away from

$$U(x) = \{x \neq 0\}$$

$$(I : x^\infty) = \{g \mid x^N g \in \overset{I}{\overset{\shortparallel}{(xy)}} \text{ for some } N\}$$

- $g = y \qquad xy \in I \implies y \in (I : x^\infty)$
- $g = y^n \qquad xy^n \in I \implies (y) \subseteq (I : x^\infty)$

Viceversa: $g \in (I : x^\infty) \implies x^N g \in (xy)$ for some N

$\implies x^N g = hxy$ for some $h \in R$

$\implies x^{N-1} g = hy \underset{\substack{\uparrow \\ \mathbb{C}[x,y] \\ \text{domain}}}{\implies} y \mid g \implies g \in (y)$

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