Kinematic Varieties for Massless Particles

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Joint work with Smita Rajan and Svala Sverrisdóttir

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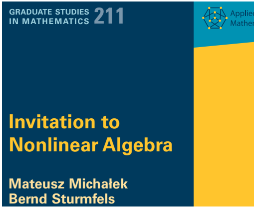
Context

This talk presents a joint paper with two Berkeley PhD students, Smita Rajan and Svala Sverrisdóttir, intended for publication in a special volume Positive Geometry in the journal *Le Matematiche*.

Initial motivation: understand the mathematics behind Smita's Bachelor thesis (Physics at Brown U), and answer a question in

A. Pokraka, S. Rajan, L. Ren, A. Volovich, W. Zhao:
Five-dimensional spinor helicity for all masses and spins
`arXiv:2405.09533`, Journal of High Energy Physics.

Another goal: Extend nonlinear algebra in
Y. El Maazouz, A. Pfister and B. Sturmfels:
Spinor-helicity varieties, `arXiv:2406.17331`.



Particles in d-dimensional spacetime

Spacetime is $\mathbb{R}^d$ or $\mathbb{C}^d$, with the Lorentzian inner product

$$x \cdot y \;=\; x_1 y_1 - x_2 y_2 - \cdots - x_n y_n.$$

The *Lorentz group* $\mathrm{SO}(1, d-1)$ consists of $d \times d$ matrices g such that $\det(g) = 1$ and $(gx) \cdot (gy) = x \cdot y$ for all $x, y \in \mathbb{C}^d$.

A configuration of n particles is given by momentum vectors

$$p_i \;=\; (p_{i1}, p_{i2}, \ldots, p_{id}) \;\in\; \mathbb{C}^d.$$

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$$p_i \;=\; (p_{i1}, p_{i2}, \ldots, p_{id}) \;\in\; \mathbb{C}^d.$$

Assume that each particle is *massless*, i.e. $p_i \cdot p_i = 0$:

$$p_{i1}^2 - p_{i2}^2 - p_{i3}^2 - \cdots - p_{id}^2 \;=\; 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Also assume *momentum conservation* $\sum_{i=1}^n p_i = 0$:

$$p_{1j} + p_{2j} + \cdots + p_{nj} \;=\; 0 \quad \text{for} \quad j = 1, 2, \ldots, d.$$

Ideals, varieties and algorithms

Let $I_{d,n} \subset \mathbb{C}[p]$ be the ideal generated by the n quadrics for massless and the d linear forms for momentum conservation.

Here $\mathbb{C}[p]$ is the polynomial ring in nd variables p_{ij}.

Example ($n = d = 3$)

Three particles on the icecream cone. Let's try it in `Macaulay2`:

i1 : R = QQ[p11,p12,p13, p21,p22,p23, p31,p32,p33];

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i2 : I = ideal( p11+p21+p31, p12+p22+p32, p13+p23+p33,
         p11^2-p12^2-p13^2, p21^2-p22^2-p23^2, p31^2-p32^2-p33^2);
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i3 : codim I, degree I

o3 = (6, 8)

i4 : isPrime I, isPrimary I

o4 = (false, true)

i5 : radical I33

o5 = ideal(. . . , $p23*p31 - p21*p33,\ p22*p31 - p21*p32$, . . .)

Prime time

Theorem

$I_{d,n}$ is prime and a complete intersection, provided $\max(n, d) \geq 4$.

Proof: technical commutative algebra

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How about using a parametric representation of the variety $V(I_n)$?

One idea is to express the variables in the first row and column in terms of the entries of the $(n-1) \times (d-1)$ matrix $p' = (p_{ij})_{i,j \geq 2}$.

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Remark (Bad News)

The elimination ideal $I_{d,n} \cap \mathbb{C}[p']$ is principal. Its generator is a large polynomial of degree 2^{n-1}. This hypersurface is a notable obstruction to any easy parametrization. This does not exist.

Example: for $n = 4, d = 5$, the polynomial has 4671 terms of degree 8.

Use Hodges' Momentum Twistors?

Mandelstam invariants

Physical properties of our n particles are invariant under the group $G = \mathrm{O}(1, d-1)$. The ring of G-invariants in $\mathbb{C}[p]$ is generated by the *Mandelstam invariants* $s_{ij} = p_i \cdot p_j$. Consider the invariant ring

$$(\mathbb{C}[p]/I_{d,n})^G \;=\; \mathbb{C}[S]/M_{d,n}.$$

The *Mandelstam variety* is the GIT quotient

$$V(M_{d,n}) \;=\; \mathrm{Spec}\big(\,(\mathbb{C}[p]/I_{d,n})^G\,\big) \;=\; V(I_{d,n})/\!/G.$$

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Theorem

Let $n \geq 2$ and $d \geq 4$. The prime ideal $M_{d,n}$ equals

$$\langle s_{11}, s_{22}, \ldots, s_{nn} \rangle + \langle \textstyle\sum_{j=1}^{n} s_{ij} \text{ for } i = 1, \ldots, n \rangle$$
$$+ \langle (d+1) \times (d+1) \text{ minors of the symmetric matrix } (s_{ij}) \rangle$$

The dimension of the Mandelstam variety is

$$\dim(V(M_{d,n})) \; = \; nd - n - d - \binom{d}{2} \; = \; \dim(V(I_{d,n})) - \dim(G).$$

Clifford algebras and spinors

We now dive into the formalism used in physics:

A. Pokraka, S. Rajan, L. Ren, A. Volovich, W. Zhao: *Five-dimensional spinor helicity for all masses and spins*, arXiv:2405.09533, JHEP.

Kinematic data for n particles are expressed in terms of spinors:

H. Elvang and Y. Huang: *Scattering Amplitudes in Gauge Theory and Gravity*, Cambridge University Press, 2015.

This encoding rests on the Clifford algebra $\mathrm{Cl}(1, d-1)$:

M. Rausch de Traubenberg: *Clifford algebras in physics*, Adv. Appl. Clifford Algebr. **19** (2009) 869–908.

Mathematicians appreciate Bourbaki:

C. Chevalley: *The Algebraic Theory of Spinors and Clifford Algebras*: Collected Works of Claude Chevalley, Volume 2, Springer Verlag, 1996.

Dirac matrices

For us, spinors are vectors of length 2^k where $k = \lfloor d/2 \rfloor$. We recursively define $2^k \times 2^k$ matrices $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$. For $d = 2$,

$$\Gamma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For larger $d = 2k$, take tensor products with Pauli matrices:

$$\Gamma_i \;=\; \Gamma_{k-1,i} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \;\text{ for }\; 1 \leq i \leq 2k-2,$$

$$\Gamma_{2k-1} \;=\; \mathrm{Id}_{2^{k-1}} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Gamma_{2k} \;=\; \mathrm{Id}_{2^{k-1}} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

For $d = 2k+1$ odd, set $\Gamma_{2k+1} \;=\; -i^{k-1} \cdot \Gamma_1 \Gamma_2 \cdots \Gamma_{2k-1}$.

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For $d = 2k+1$ odd, set $\Gamma_{2k+1} = -i^{k-1} \cdot \Gamma_1 \Gamma_2 \cdots \Gamma_{2k-1}$.

Proposition

The Dirac matrices satisfy the Clifford algebra relations:

$$\Gamma_1^2 = -2\,\mathrm{Id}_{2^k}, \ \Gamma_j^2 = 2\,\mathrm{Id}_{2^k} \text{ for } j \geq 2$$
$$\text{and } \Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0_{2^k} \text{ for } i \neq j.$$

Cl(1, d−1)

A matrix for one particle

The *momentum space Dirac matrix* is the linear combination

$$P \;=\; -p_1\Gamma_1 + p_2\Gamma_2 + p_3\Gamma_3 + \cdots + p_d\Gamma_d.$$

Example $(d = 4, 5, 6)$

$$P = \begin{bmatrix} 0 & 0 & p_1 - p_2 & p_3 - ip_4 \\ 0 & 0 & p_3 + ip_4 & p_1 + p_2 \\ -p_1 - p_2 & p_3 - ip_4 & 0 & 0 \\ p_3 + ip_4 & -p_1 + p_2 & 0 & 0 \end{bmatrix},$$

$$P \;=\; \begin{bmatrix} p_5 & 0 & p_1 - p_2 & p_3 - ip_4 \\ 0 & p_5 & p_3 + ip_4 & p_1 + p_2 \\ -p_1 - p_2 & p_3 - ip_4 & -p_5 & 0 \\ p_3 + ip_4 & -p_1 + p_2 & 0 & -p_5 \end{bmatrix}.$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & -p_1 + p_2 & 0 & -p_3 + ip_4 & p_5 - ip_6 \\ 0 & 0 & 0 & 0 & 0 & -p_1 + p_2 & p_5 + ip_6 & p_3 + ip_4 \\ 0 & 0 & 0 & 0 & -p_3 - ip_4 & p_5 - ip_6 & -p_1 - p_2 & 0 \\ 0 & 0 & 0 & 0 & p_5 + ip_6 & p_3 - ip_4 & 0 & -p_1 - p_2 \\ p_1 + p_2 & 0 & -p_3 + ip_4 & p_5 - ip_6 & 0 & 0 & 0 & 0 \\ 0 & p_1 + p_2 & p_5 + ip_6 & p_3 + ip_4 & 0 & 0 & 0 & 0 \\ -p_3 - ip_4 & p_5 - ip_6 & p_1 - p_2 & 0 & 0 & 0 & 0 & 0 \\ p_5 + ip_6 & p_3 - ip_4 & 0 & p_1 - p_2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Spin representation

Corollary

The relations of the Clifford algebra $\mathrm{Cl}(1, d-1)$ *imply*

$$\begin{aligned} P^2 &= (-p_1^2 + p_2^2 + \cdots + p_d^2)\,\mathrm{Id}_{2^k}, \\ \det(P) &= (p_1^2 - p_2^2 - \cdots - p_d^2)^{2^{k-1}}. \end{aligned}$$

For massless particles, the momentum space Dirac matrix P is nilpotent and its rank equals half of its size, i.e. $\mathrm{rank}(P) = 2^{k-1}$.

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The Dirac representation of $\mathrm{Cl}(1, d-1)$ gives rise to the spin representation of the Lie algebra $\mathfrak{so}(1, d-1)$. The commutators

$$\Sigma_{jk} = \frac{1}{4}[\Gamma_j, \Gamma_k]$$

satisfy same relations as the generators of $\mathfrak{so}(1, d-1)$.

The spin representation of $\mathrm{SO}(1, d-1)$ is the action of the matrix exponentials $\exp(\Sigma_{jk})$ on spinor space $\mathbb{C}^{2^k}$.

Charge conjugation matrix

An equivariant linear map from the spin representation of $\mathfrak{so}(1, d-1)$ to its dual is represented by a $2^k \times 2^k$ matrix C:

$$\begin{aligned} CP &= -P^T C && \text{if } d = 2k \text{ is even,} \\ CP &= (-1)^k P^T C && \text{if } d = 2k+1 \text{ is odd.} \end{aligned}$$

Example $(d = 4, 5, 6)$

$$C = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Proposition (Symmetries)

1. *C is symmetric for $k \equiv 0, 3 \bmod 4$, otherwise skew symmetric.*
2. *C is block diagonal for $k \equiv 0 \bmod 2$, else anti-block diagonal.*
3. *the $2^{k-1} \times 2^{k-1}$ blocks of C are skew symmetric when $k = 2, 3 \bmod 4$; otherwise the blocks are symmetric.*

Bra and ket

Our goal: model interactions among n massless particles

$p_i = (p_{i1}, \dots, p_{id})$. The tuple $(p_1, \dots, p_n)$ lies in $V(I_{d,n}) \subset \mathbb{C}^{nd}$.

The *momentum space Dirac matrix* for the ith particle is

$$P_i \;=\; -p_{i1}\Gamma_1 + p_{i2}\Gamma_2 + p_{i3}\Gamma_3 + \cdots + p_{id}\Gamma_d.$$

This matrix has size 2^k and rank 2^{k-1}. Clifford relations imply

$$P_iP_j + P_jP_i \;=\; 2p_i \cdot p_j \,\mathrm{Id}_{2^k} \;=\; 2s_{ij}\,\mathrm{Id}_{2^k}.$$

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We parameterize the column space of P_i using a vector
$z_i = (\, z_{i,1}, z_{i,2}, \ldots, z_{i,2^{k-2}}, 0, 0, \ldots, 0\, , z_{i,2^{k-2}+1}, \ldots, z_{i,2^{k-1}}\,)^T$.

Use Dirac's ket-notation for vectors in this column space:

$$|\, i \,\rangle \;=\; P_i\, z_i.$$

Use the bra-notation $\langle\, i \,|$ for the row vector $|\, i \,\rangle^T$. The spinors $|\, i \,\rangle$ and $\langle\, i \,|$ depend on $d + 2^{k-1}$ parameters. They represent particle i.

Spinor brackets

The *spinor brackets* of order two and three are

$$\langle\, ij \,\rangle \;=\; \langle\, i \,|\, C \,|\, j \,\rangle \quad \text{and} \quad \langle\, ij\, k \,\rangle \;=\; \langle\, i \,|\, CP_j \,|\, k \,\rangle.$$

Here $i, j, k \in \{1, 2, \ldots, n\}$. The *$\ell$-th order spinor brackets* are

$$\langle\, i_1 i_2 \,\cdots\, i_\ell \,\rangle \;=\; \langle\, i_1 \,|\, CP_{i_2} \cdots P_{i_{\ell-1}} \,|\, i_\ell \,\rangle.$$

Spinor brackets are Lorentz-invariant elements in the ring

$$R_{d,n} \;=\; \mathbb{C}[p, z]/I_{d,n},$$

which is generated by nd parameters p_{ij} and $n2^{k-1}$ parameters z_{ij}.

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Example. For $d = 3$ we have

$$\begin{aligned}
\langle ij \rangle &= \begin{bmatrix} z_{i1} & 0 \end{bmatrix} \begin{bmatrix} p_{i3} & p_{i1}+p_{i2} \\ -p_{i1}+p_{i2} & -p_{i3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_{j3} & -p_{j1}+p_{j2} \\ p_{j1}+p_{j2} & -p_{j3} \end{bmatrix} \begin{bmatrix} z_{j1} \\ 0 \end{bmatrix} \\
&= -p_{i1}p_{j3}z_{i1}z_{j1} - p_{i2}p_{j3}z_{i1}z_{j1} + p_{i3}p_{j1}z_{i1}z_{j1} + p_{i3}p_{j2}z_{i1}z_{j1}, \\
\langle ijk \rangle &= p_{i1}p_{j1}p_{k1}z_{i1}z_{k1} + p_{i1}p_{j1}p_{k2}z_{i1}z_{k1} - p_{i1}p_{j2}p_{k1}z_{i1}z_{k1} - p_{i1}p_{j2}p_{k2}z_{i1}z_{k1} \\
&\quad - p_{i1}p_{j3}p_{k3}z_{i1}z_{k1} + p_{i2}p_{j1}p_{k1}z_{i1}z_{k1} + p_{i2}p_{j1}p_{k2}z_{i1}z_{k1} - p_{i2}p_{j2}p_{k1}z_{i1}z_{k1} \\
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&\quad - p_{i3}p_{j3}p_{k1}z_{i1}z_{k1} - p_{i3}p_{j3}p_{k2}z_{i1}z_{k1}.
\end{aligned}$$

Matrices of spinor brackets

Multiply matrices of formats $n \times 2^k$, $2^k \times 2^k$ and $2^k \times n$ to define

$$S \;:=\; \big(\langle\, ij \,\rangle\big)_{1\le i,j\le n} \;=\; \big(|\,1\,\rangle,\ldots,|\,n\,\rangle\big)^T \cdot C \cdot \big(|\,1\,\rangle,\ldots,|\,n\,\rangle\big).$$

$$T_j \;:=\; \big(\langle\, ijk \,\rangle\big)_{1\le i,k\le n} \;=\; \big(|\,1\,\rangle,\ldots,|\,n\,\rangle\big)^T \cdot C \cdot P_j \cdot \big(|\,1\,\rangle,\ldots,|\,n\,\rangle\big).$$

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Theorem

The $n \times n$ matrix S has rank $\le 2^k$ with zeros on the diagonal. If $k \equiv 0, 3 \bmod 4$ then S is symmetric; otherwise skew symmetric:

$$\langle\, i\, i \,\rangle \;=\; 0 \quad \text{and} \quad \langle\, i\, j \,\rangle \;=\; \pm\langle\, j\, i \,\rangle \qquad \textit{for } 1 \le i,j \le n.$$

The matrix T_j has rank $\le 2^{k-1}$ with zeros row and column j. If $d \equiv 1,2,3,4 \bmod 8$ then T_j is symmetric; else skew symmetric:

$$\langle\, j\, j\, k \,\rangle \;=\; \langle\, i\, j\, j \,\rangle \;=\; 0 \;\text{ and }\; \langle\, i\, j\, k \,\rangle \;=\; \pm\langle\, k\, j\, i \,\rangle \quad \textit{for } 1 \le i,j,k \le n.$$

The sum of the matrices T_j is zero: $T_1 + T_2 + \cdots + T_n = 0$.

Example: Four particles for flatlanders

For $d = 3, k = 1, n = 4$, there are six order two spinor brackets:

$$S = \begin{bmatrix} 0 & \langle 12\rangle & \langle 13\rangle & \langle 14\rangle \\ -\langle 12\rangle & 0 & \langle 23\rangle & \langle 24\rangle \\ -\langle 13\rangle & -\langle 23\rangle & 0 & \langle 34\rangle \\ -\langle 14\rangle & -\langle 24\rangle & -\langle 34\rangle & 0 \end{bmatrix}.$$

The 24 spinor brackets of order three are the entries of

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \langle 212\rangle & \langle 213\rangle & \langle 214\rangle \\ 0 & \langle 213\rangle & \langle 313\rangle & \langle 314\rangle \\ 0 & \langle 214\rangle & \langle 314\rangle & \langle 414\rangle \end{bmatrix}, \quad T_2 = \begin{bmatrix} \langle 121\rangle & 0 & \langle 123\rangle & \langle 124\rangle \\ 0 & 0 & 0 & 0 \\ \langle 123\rangle & 0 & \langle 323\rangle & \langle 324\rangle \\ \langle 124\rangle & 0 & \langle 324\rangle & \langle 424\rangle \end{bmatrix},$$

$$T_3 = \begin{bmatrix} \langle 131\rangle & \langle 132\rangle & 0 & \langle 134\rangle \\ \langle 132\rangle & \langle 232\rangle & 0 & \langle 234\rangle \\ 0 & 0 & 0 & 0 \\ \langle 134\rangle & \langle 234\rangle & 0 & \langle 434\rangle \end{bmatrix}, \quad T_4 = \begin{bmatrix} \langle 141\rangle & \langle 142\rangle & \langle 143\rangle & 0 \\ \langle 142\rangle & \langle 242\rangle & \langle 243\rangle & 0 \\ \langle 143\rangle & \langle 243\rangle & \langle 343\rangle & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example: Four particles for flatlanders

The 30 brackets define the kinematic variety in $\mathbb{P}^5 \times \mathbb{P}^{23}$.
This is irreducible of dimension 4 and its multidegree is

$$5s^5t^{19} + 28s^4t^{20} + 24s^3t^{21} + 10s^2t^{22} + 2st^{23} \in H^*(\mathbb{P}^5 \times \mathbb{P}^{23}, \mathbb{Z}).$$

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The ideal is generated by 10 linear forms in $T_1 + T_2 + T_3 + T_4$ plus $54 = 1 + 24 + 29$ quadrics. The Plücker quadric

$$\langle 12\rangle\langle 34\rangle - \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle \quad = \quad \mathrm{Pfaffian}(S),$$

ensures that S has rank two. The 24 binomial quadrics

$$\langle i\,j\,k\rangle\langle l\,j\,m\rangle - \langle i\,j\,m\rangle\langle l\,j\,k\rangle.$$

are 2×2 minors of the slices T_j. Finally, 29 bilinear relations like

$$\langle 12\rangle\langle 324\rangle - \langle 34\rangle\langle 142\rangle \quad \text{and} \quad \langle 12\rangle\langle 243\rangle - \langle 13\rangle\langle 242\rangle + \langle 23\rangle\langle 142\rangle$$

ensure that the $4 \times 4 \times 5$ tensor ST has rank two. They are in the radical of the 3×3 minors of the 4×20 matrix (S, T_1, T_2, T_3, T_4).

Varieties in matrix space

The $\binom{n}{2}$ order two spinor brackets $\langle ij \rangle$ form an $n \times n$ matrix S which is either symmetric or skew symmetric. The set of all such matrices defines the *kinematic variety* $\mathcal{K}^{(2)}_{d,n}$ in $\mathbb{P}^{\binom{n}{2}-1}$.

Theorem

For $d = 3$, the ideal of $\mathcal{K}^{(2)}_{3,n}$ is given by 4×4-Pfaffians of a skew $n \times n$ matrix, so it is the Grassmannian $\mathrm{Gr}(2, n)$. For $d = 4, 5$, we get 6×6-Pfaffians, so $\mathcal{K}^{(2)}_{d,n}$ is the secant variety of $\mathrm{Gr}(2, n)$.

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Conjecture

For $d = 6, 7, 8, 9$, the kinematic variety $\mathcal{K}_{d,n}^{(2)}$ consists of all symmetric $n \times n$ matrices with zero diagonal and rank $\leq 2^{\lfloor d/2 \rfloor}$.

For d even, the spin representation splits into two irreducibles.
Use separate brackets $\langle ij \rangle$ and $[ij]$ for each block.

For $d = 4$, we recover the flag varieties $\mathrm{Fl}(2, n-2; \mathbb{C}^n)$ in

Y. El Maazouz, A. Pfister and BSt: *Spinor-helicity varieties*, 2024.

Varieties in tensor space

Write $\mathcal{K}^{(3)}_{d,n}$ for the kinematic variety of $n \times n \times (n+1)$ tensors ST.
The $n \times n$ slices $S, T_1, \ldots, T_n$ are symmetric or skew symmetric, depending on residue classes of $k = \lfloor d/2 \rfloor$ mod 4 and d mod 8.

The ideal of $\mathcal{K}^{(3)}_{d,n}$ is $\mathbb{Z}^2$-graded.

The variety lives in $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{K-1}$, where

- $K = n \cdot \binom{n}{2}$ when slices T_j are symmetric $(d \equiv 1, 2, 3, 4 \bmod 8)$,
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Conjecture (Flatlanders)

The variety $\mathcal{K}_{3,n}^{(3)}$ has dimension $3n - 8$. Its points are tensors ST of rank 2, where S is skew symmetric and the T_j are symmetric of rank ≤ 1, summing to 0, with zeros in j-th row/column.

Its ideal is generated by linear forms and quadrics:
the entries of $T_1 + \cdots + T_n$, the 4×4 pfaffians of S,
the 2×2 minors of the T_j, and bilinear Pfaffians in the
radical of the 3×3 minors of the flattening $(S, T_1, \ldots, T_n)$.

Numerical algebraic geometry

Goal: Study the kinematic varieties $\mathcal{K}^{(3)}_{d,n}$ for arbitrary d and n.
First question: What is the dimension?

Proposition

The dimensions of small kinematic varieties $\mathcal{K}^{(3)}_{d,n}$ *are*

$d \backslash n$	4	5	6	7	8	9	10	11	12
4	8	13	18	23	28	33	38	43	48
5	7	**13**	19	25	31	37	43	49	55
6	9	20	30	40	49	58	67	76	85
7	9	20	30	40	50	60	70	80	90
8	10	28	51	67	82	97	112	127	142
9	15	33	49	65	81	97	113	129	145

We computed these numbers with numerical software in `julia`:
P. Breiding and S. Timme: *HomotopyContinuation.jl: A package for homotopy continuation in Julia*, Mathematical Software, ICMS 2018, Lecture Notes in Computer Science, **10931**, 458-465, 2018.

Five-dimensional spacetime

Points in $\mathcal{K}_{5,n}^{(3)}$ are $n \times n \times (n+1)$ tensors ST. Slices S and T_j are skew symmetric of rank 4 resp. 2. The T_j satisfy linear constraints.

Proposition

For each index $j \in \{1, \ldots, n\}$, the skew symmetric $n \times n$ matrix

$$\big(|1\rangle, \ldots, z_j, \ldots, |n\rangle\big)^T \cdot C \cdot P_j \cdot \big(|1\rangle, \ldots, z_j, \ldots, |n\rangle\big)$$

contains both brackets $\langle ij \rangle$ and $\langle ijk \rangle$. It has rank ≤ 2 on $\mathcal{K}_{5,n}^{(3)}$, so the 4×4 Pfaffians give bilinear ideal generators. Furthermore, the $n \times (n^2 + n)$ flattening $(S, T_1, \ldots, T_n)$ has rank ≤ 4 on $\mathcal{K}_{5,n}^{(3)}$. It contributes mixed 6×6 Pfaffians to the ideal generators.

The $n \times (n+1)$ slices of ST given by fixing indices i or k seem to have rank ≤ 3. Interestingly, *the tensor rank of ST is at least 5* on $\mathcal{K}_{5,n}^{(3)}$. We show this by evaluating the Strassen invariant on $3 \times 3 \times 3$ subtensors.

Example: Five particles in five-dim'l spacetime

The variety $\mathcal{K}_{5,5}^{(3)} \subset \mathbb{P}^9 \times \mathbb{P}^{29}$ has dimension 13. Its ideal is generated by 10 linear forms, 25 quadrics, 15 cubics and 5 quartics. Each T_j is a skew symmetric with a zero row, so it contributes one Pfaffian $\langle ijk\rangle\langle \ell jm\rangle - \langle ij\ell\rangle\langle kjm\rangle + \langle ijm\rangle\langle kj\ell\rangle$.

The other 20 quadrics are bilinear, e.g. five 4×4 Pfaffians of

$$\begin{bmatrix} 0 & \langle 12\rangle & \langle 13\rangle & \langle 14\rangle & \langle 15\rangle \\ -\langle 12\rangle & 0 & \langle 213\rangle & \langle 214\rangle & \langle 215\rangle \\ -\langle 13\rangle & -\langle 213\rangle & 0 & \langle 314\rangle & \langle 315\rangle \\ -\langle 14\rangle & -\langle 214\rangle & -\langle 314\rangle & 0 & \langle 415\rangle \\ -\langle 15\rangle & -\langle 215\rangle & -\langle 315\rangle & -\langle 415\rangle & 0 \end{bmatrix}.$$

The 15 cubics ensure that $(S, T_1, T_2, T_3, T_4, T_5)$ has rank ≤ 4. One of them is $\langle 213\rangle\langle 123\rangle\langle 435\rangle - \langle 213\rangle\langle 325\rangle\langle 134\rangle + \langle 213\rangle\langle 324\rangle\langle 135\rangle +$ $\langle 314\rangle\langle 123\rangle\langle 235\rangle - \langle 314\rangle\langle 325\rangle\langle 132\rangle - \langle 315\rangle\langle 123\rangle\langle 234\rangle + \langle 315\rangle\langle 324\rangle\langle 132\rangle$.

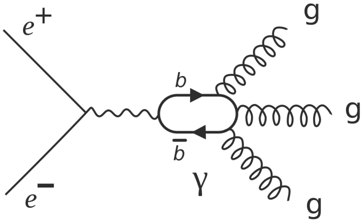
The 5 quartics are 4×4 minors of mixed slices like

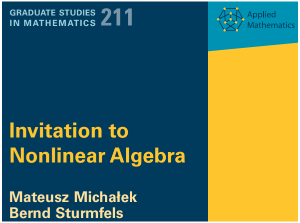
$$\begin{bmatrix} 0 & \langle 12\rangle & \langle 13\rangle & \langle 14\rangle & \langle 15\rangle \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \langle 123\rangle & \langle 124\rangle & \langle 125\rangle \\ 0 & \langle 132\rangle & 0 & \langle 134\rangle & \langle 135\rangle \\ 0 & \langle 142\rangle & \langle 143\rangle & 0 & \langle 145\rangle \\ 0 & \langle 152\rangle & \langle 153\rangle & \langle 154\rangle & 0 \end{bmatrix}, \quad \begin{bmatrix} -\langle 12\rangle & 0 & \langle 23\rangle & \langle 24\rangle & \langle 25\rangle \\ 0 & 0 & \langle 213\rangle & \langle 214\rangle & \langle 215\rangle \\ 0 & 0 & 0 & 0 & 0 \\ -\langle 132\rangle & 0 & 0 & \langle 234\rangle & \langle 235\rangle \\ -\langle 142\rangle & 0 & \langle 243\rangle & 0 & \langle 245\rangle \\ -\langle 152\rangle & 0 & \langle 253\rangle & \langle 254\rangle & 0 \end{bmatrix}, \ \mathrm{etc}\ldots$$

Conclusion

I learned a lot from Smita Rajan and Svala Sverrisdóttir. We hope you'll enjoy the papers on **Positive Geometry** in *Le Matematiche*.

Many connections remain to be explored:





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