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Erik Panzer

“Tropical quantum field theory and asymptotics of perturbation theory”

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Tropical quantum field theory and asymptotics of perturbation theory

Erik Panzer (Oxford)

Royal Society University Research Fellow

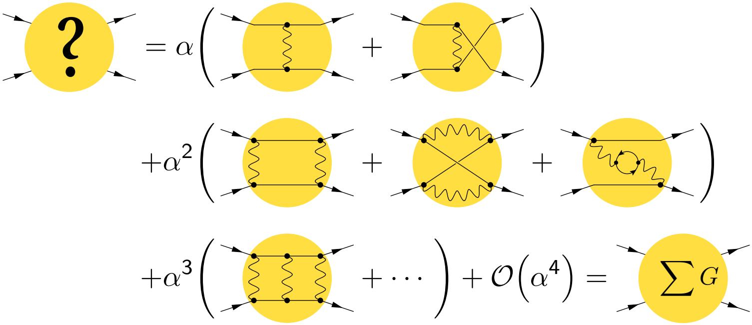
arXiv:2512.21091

with **Paul-Hermann Balduf**

8 April 2026

Universe+ Online Seminar

Quantum Field Theory – perturbative



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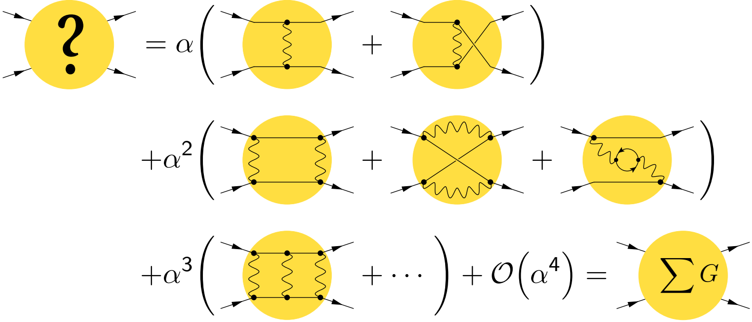

Feynman integral

$$\mathcal{I}_G(D; \{\vec{p}_i \cdot \vec{p}_j, m_e^2\}) = \prod_i \int_{\mathbb{R}^D} \mathrm{d}^D k_i \prod_e \frac{1}{q_e^2 + m_e^2}$$

Amplitude $= \sum_G \alpha^n \mathcal{I}_G$ compute more ➡ higher accuracy

Problems:

1. $\mathcal{I}_G$ extremely complicated

   ➡ *combinatorial explosion, integration by parts, polylogarithms, modular forms, Calabi-Yau manifolds, …*

2. $\sum_G \alpha^n \mathcal{I}_G = \infty$

   ➡ *factorial growth $C \cdot n! \cdot b^n \cdot n^\alpha$, resummation, Borel transformation, resurgence, …*

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Simplifications:

1. integrable models, large N

   ➡ *exact $\sum_G \alpha^n \mathcal{I}_G$, but typically convergent*

2. truncated Dyson-Schwinger equations

   ➡ *factorial growth, tiny class of diagrams*

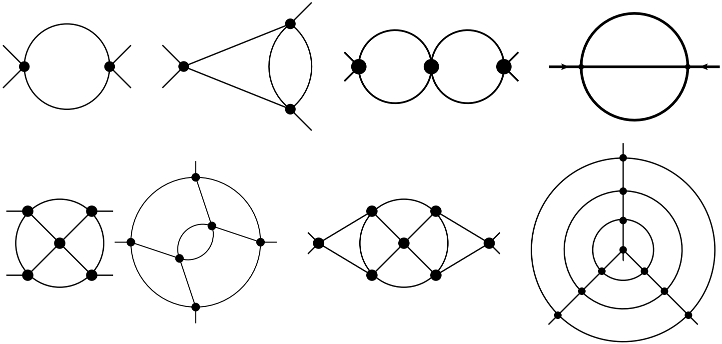
3. tropical limit

   ➡ *all $\mathcal{I}_G$ simplify drastically, no truncation*

What is known?

ϕ^4 **theory:**

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} m^2 \phi^2 + \frac{16\pi^2}{4!} g \phi^4$$



[g^6: 1993, g^7: Kompaniets–Panzer 2016, g^8: Schnetz 2018]

$$\beta^{\mathrm{MS}}(g) = \sum_k \beta_k^{\mathrm{MS}}(-g)^k \approx 3g^2 - 5.667g^3 + 32.55g^4 - 271.6g^5 + 2848.6g^6 - 34776g^7 + 474651g^8 + \mathcal{O}(g^9)$$

Lipatov & Brézin–Le Guillou–Zinn-Justin 1976, McKane–Wallace–Bonfim 1984

$$\beta_k^{\mathrm{MS}} \sim \overline{\beta}_k := k! \cdot k^{7/2} \cdot C \qquad C = \frac{144}{\pi^{3/2}} e^{-\frac{17}{4} - 3\gamma_E + 6\zeta'(-1)} \approx 0.0242$$

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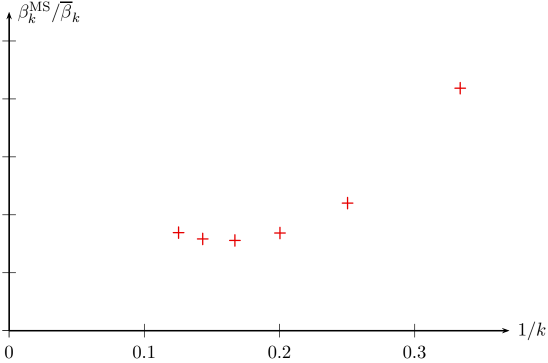

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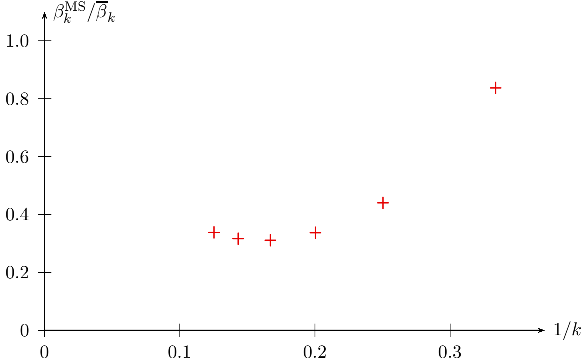

Borel resummation:

1. Series to be resummed:

$$f(g) = \sum_k c_k g^k$$

2. Borel transform:

$$\mathfrak{B}(u) = \sum_k \frac{c_k}{k!} u^k$$

3. Analytic continuation of $\mathfrak{B}$ from small $|u|$ to all $u \in \mathbb{R}_{\geq 0}$

4. Inverse Borel transform:

$$\widetilde{f}(g) = \int_0^\infty \mathfrak{B}(ug) e^{-u} \mathrm{d}u$$

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   ⇨ problematic (singularities?)

4. Inverse Borel transform:

$$\widetilde{f}(g) = \int_0^\infty \mathfrak{B}(ug) e^{-u} \mathrm{d}u$$

   ⇨ problematic (convergence?)

McKane–Wallace–Bonfim

The Borel transform of $f = \beta^{\mathrm{MS}}$ is analytic for $|u| < 1$ and has a singularity at $u = -1$ (first instanton).

long-range theory

bound/approximation

combinatorial rule

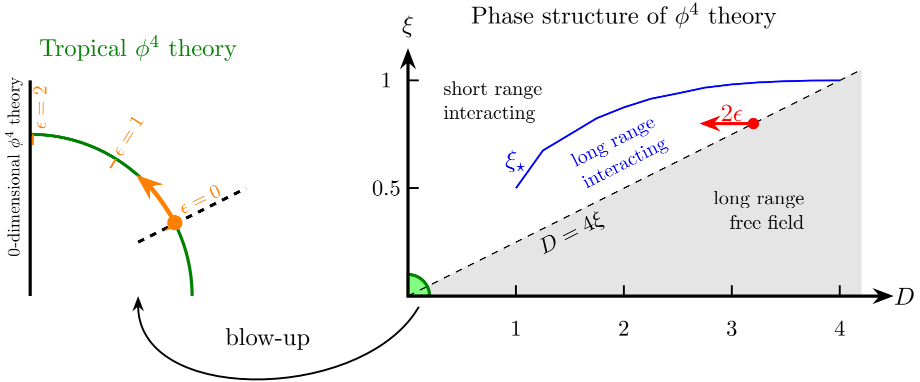
The tropical limit

$$\lim_{\xi \to 0} \int_{\mathbb{R}^{\xi(4-2\varepsilon)}} \left[\frac{1}{2}\phi(-\partial_\mu \partial^\mu + m^2)^\xi \phi + \frac{16\pi^2}{4!} g \phi^4 \right]$$

long-range theory — bound/approximation — combinatorial rule

The tropical limit

$$\lim_{\xi\to 0}\int_{\mathbb{R}^{\xi(4-2\varepsilon)}}\left[\frac{1}{2}\phi(-\partial_\mu\partial^\mu+m^2)^\xi\phi+\frac{16\pi^2}{4!}g\phi^4\right]$$



Feynman integral in momentum space:

$$\mathcal{I}_G^{\mathrm{tr}}(D; \{\vec{p}_i \cdot \vec{p}_j, m^2\}) = \lim_{\xi \to 0} \left[\prod_{i=1}^{h_1} \int_{\mathbb{R}^{\xi D}} \frac{\mathrm{d}^{\xi D} k_i}{\pi^{\xi D/2}} \prod_{e=1}^{E} \frac{1}{(q_e^2 + m^2)^{\xi}} \right]$$

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Feynman representaton:

$$\lim_{\xi \to 0} \left[\Gamma(\xi \omega_G) \prod_{e=1}^{E-1} \int_0^{\infty} \frac{x_e^{\xi - 1} \mathrm{d} x_e}{\Gamma(\xi)} \frac{1}{\mathcal{U}^{\xi D/2}} \left(\frac{\mathcal{U}}{\mathcal{F}} \right)^{\xi \omega_G} \right]$$

Superficial degree of convergence:

$$\omega_G = |E(G)| - h_1(G) \frac{D}{2}$$

Symanzik polynomials:

$$\mathcal{U} = \sum_T \prod_{e \notin T} x_e \qquad \mathcal{F} = m^2 (\sum_e x_e) \mathcal{U} + \sum_{T \sqcup T'} (\sum_{v \in T} p_v)^2 \prod_{e \notin T \sqcup T'} x_e$$

$$y_e = x_e^{\xi} \qquad y^k = \prod_e y_e^{k_e}$$

Tropical limit

$$\lim_{\xi = 0} \left(\sum_k c_k y^{k/\xi} \right)^{\xi} = \max_{c_k \neq 0} y^k$$

$$\mathcal{U}^{\mathrm{tr}} = \max_T \prod_{e \notin T} x_e \quad \Rightarrow \quad \mathcal{U}^{\mathrm{tr}} \leq \mathcal{U} \leq \mathcal{U}^{\mathrm{tr}} \cdot |\mathrm{trees}|$$

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Tropical Feynman integral (massive)

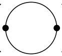
$$\mathcal{I}_G^{\mathrm{tr}}(D) = \frac{1}{\omega_G} \mathcal{H}(G) \qquad \mathcal{H}(G) = \left(\prod_e \int_0^{\infty} \mathrm{d}x_e \right) \frac{\delta(1 - \max_e x_e)}{(\mathcal{U}^{\mathrm{tr}})^{D/2}} \in \mathbb{Q}(D)$$

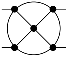
$$\operatorname*{Res}_{\omega_G=0} \mathcal{I}_G^{\mathrm{tr}} = \mathcal{H}(G) \qquad \operatorname*{Res}_{\omega_G=0} \mathcal{I}_G = \mathcal{P}(G) = \left(\prod_e \int_0^{\infty} \mathrm{d}x_e \right) \frac{\delta(1 - \max_e x_e)}{\mathcal{U}^{D/2}}$$

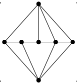
- $D = 4 - 2\varepsilon$
- log-divergent: $E = 2h_1$ ⇒
- no subdivergences

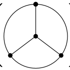
Feynman periods

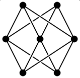
$$\mathcal{I}_G(D; \{\vec{p}_i \cdot \vec{p}_j, m_e^2\}) = \frac{\mathcal{P}(G)}{\varepsilon \cdot h_1} + \mathcal{O}\left(\varepsilon^0\right)$$

$\mathcal{P}\Big($  $\Big) = 1$

$\mathcal{P}\Big($  $\Big) = 20\zeta(5)$

$\mathcal{P}\Big($  $\Big) = \frac{1063}{9}\zeta(9) + 8\zeta(3)^3$

$\mathcal{P}\Big($  $\Big) = 6\zeta(3)$

$\mathcal{P}\Big($  $\Big) = \frac{288}{5}\Big(58\zeta(8) - 24\zeta(3,5) - 45\zeta(3)\zeta(5)\Big)$

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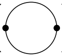
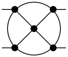
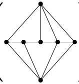
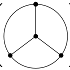
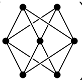
$\mathcal{P}\Big($  $\Big) = 1$

$\mathcal{P}\Big($  $\Big) = 20\zeta(5)$

$\mathcal{P}\Big($  $\Big) = \frac{1063}{9}\zeta(9) + 8\zeta(3)^3$

$\mathcal{P}\Big($  $\Big) = 6\zeta(3)$

$\mathcal{P}\Big($  $\Big) = \frac{288}{5}\big(58\zeta(8) - 24\zeta(3,5) - 45\zeta(3)\zeta(5)\big)$

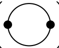
- β function, running coupling, critical exponents
- $>$ 1000 periods known [Broadhurst '85, &Kreimer '95] [Schnetz '08, &Panzer '16]
- multiple polylogarithms at 2nd, 4th, and 6th roots of unity
- modular K3 surfaces [Brown&Schnetz '10]
- modular Calabi-Yau 3-folds [Logan '16]

Hepp bound ($D = 4$)

arXiv:1908.09820

$$\mathcal{H}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{(\mathcal{U}^{\mathrm{tr}})^2|_{x_1=1}}$$

$$\mathcal{U}^{\mathrm{tr}} = \max_T \prod_{e \notin T} x_e$$

$\mathcal{H}\Big($  $\Big)$

$$= \int_0^\infty \frac{\mathrm{d}x_2}{(\max\{1, x_2\})^2}$$

$$= \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2}$$

$$= 2$$

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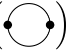
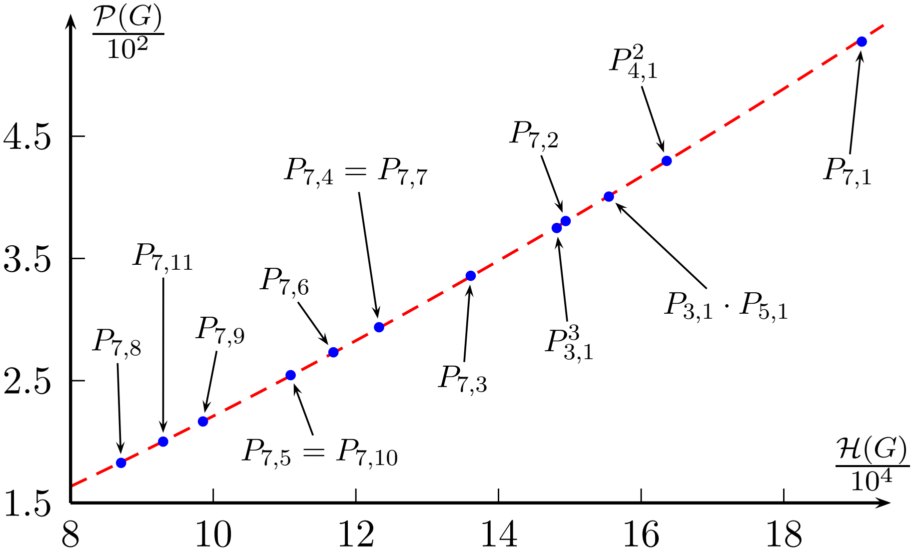
$\mathcal{H}\Big($  $\Big)$

$$= \int_0^\infty \frac{\mathrm{d}x_2}{(\max\{1, x_2\})^2} = \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2} = 2$$

- $\mathcal{H}(G) > \mathcal{P}(G) > \dfrac{\mathcal{H}(G)}{|\mathrm{trees}(G)|^2}$
- easy to compute
- same identities
- very strong correlation

G	$\mathcal{P}(G \setminus v)$	$\mathcal{H}(G \setminus v)$
$P_{7,1}$	527.7	190952
$P_{4,1} \cdot P_{4,1}$	430.1	163592
$P_{3,1} \cdot P_{5,1}$	400.9	155484
$P_{7,2}$	380.9	149426
$P_{3,1} \cdot P_{3,1} \cdot P_{3,1}$	375.2	148176
$P_{7,3}$	336.1	136114
$\{P_{7,4}, P_{7,7}\}$	294.0	123260
$P_{7,6}$	273.5	116860
$\{P_{7,5}, P_{7,10}\}$	254.8	110864
$P_{7,9}$	216.9	98568
$P_{7,11}$	200.4	92984
$P_{7,8}$	183.0	87088



➡ efficient numerics (tropical Monte Carlo) for large Feynman integrals
[Borinsky]

Minimal subtraction

- dimensional regularisation $D = 4 - 2\varepsilon$
- multiplicative renormalization
- graph-by-graph counterterms from forest formula

$$Z_G^{\mathrm{tr}} = -\mathsf{PolePart}\left(\mathcal{I}_G^{\mathrm{tr}} + \sum_{\gamma \subset G} Z_\gamma \mathcal{I}_{G/\gamma}^{\mathrm{tr}}\right) \in \mathbb{Q}\left[\frac{1}{\varepsilon}\right]$$

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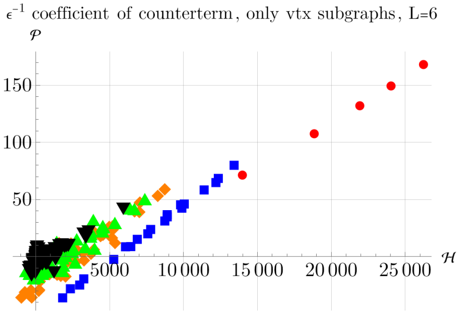
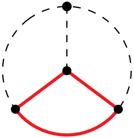
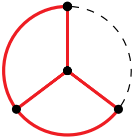
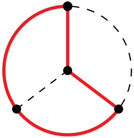
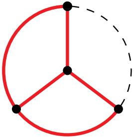
➡ correlation tropical/non-tropical counterterms:

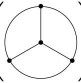


$D = 4$ $\omega(\gamma_\ell) = |E(\gamma_\ell)| - 2\ell$

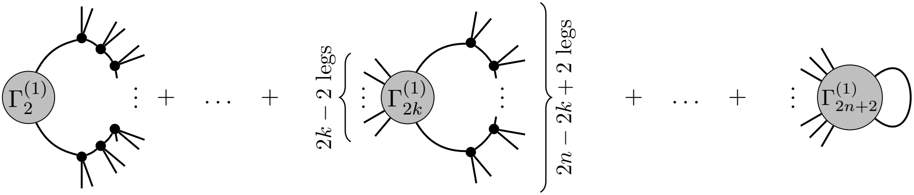
Combinatorial formula

$$\mathcal{H}(G) = \sum_{\substack{\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_\ell = G \\ \text{each } \gamma_i \text{ is 1PI}}} \frac{|\gamma_1| \cdot |\gamma_2 \setminus \gamma_1| \cdots |G \setminus \gamma_{\ell-1}|}{\omega(\gamma_1) \cdots \omega(\gamma_{\ell-1})}$$

γ_1	$\subset$	γ_2	summand	#	$\sum$
	$\subset$		$\frac{3\cdot 2\cdot 1}{1\cdot 1} = 6$	12	72
	$\subset$		$\frac{4\cdot 1\cdot 1}{2\cdot 1} = 2$	6	12

$\Rightarrow \mathcal{H}\Big($  $\Big) = 84$

Recurrence relations (example $\Gamma_n^{(2)}$):



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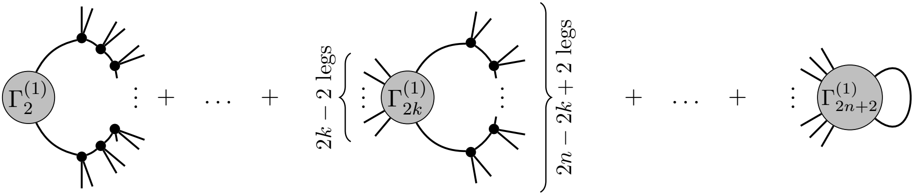


Differential equation/recurrence relations (Borinsky)

$$(2\varepsilon t\partial_t + x\partial_x - 4)\mathcal{G}(x,t) = t\cdot\left(\frac{1}{1-\partial_x^2\mathcal{G}(x,t)} - 1\right)$$

$$\mathcal{G}(x,t) = \sum_{n,\ell\geq 0}\Gamma_n^{(\ell)} t^\ell \frac{x^n}{n!}$$

- efficient computation to hundreds of loops
- version for $O(N)$ symmetry (vector model)
- same pde in local potential approximation [Felder '87]

Asymptotics of the tropical beta function

$$\beta^{\mathrm{tr,MS}} = -2\epsilon g + 6g^2 - 36g^3 + 522g^4 - 11256g^5 + \frac{1224063}{4}g^6 - \frac{97292007}{10}g^7 + \dots$$

➜ 400 terms computed

Series analysis & differential approximants

$$\beta_n^{\mathrm{tr,MS}} \sim C \cdot n! \cdot n^{5/2} \cdot (-3)^n \left(1 + \qquad \right)$$

➜ first instanton at $u = -1/3$

$$\beta^{\mathrm{tr,MS}} = -2\epsilon g + 6g^2 - 36g^3 + 522g^4 - 11256g^5 + \frac{1224063}{4}g^6 - \frac{97292007}{10}g^7 + \dots$$

➡ 400 terms computed

Series analysis & differential approximants

$$\beta_n^{\mathrm{tr,MS}} \sim C \cdot n! \cdot n^{5/2} \cdot (-3)^n \left(1 + \frac{c_1}{n^{1/3}} + \frac{c_2}{n^{2/3}} + \frac{c_3}{n} + \frac{c_4}{n}\log n + \dots\right)$$

➡ first instanton at $u = -1/3$

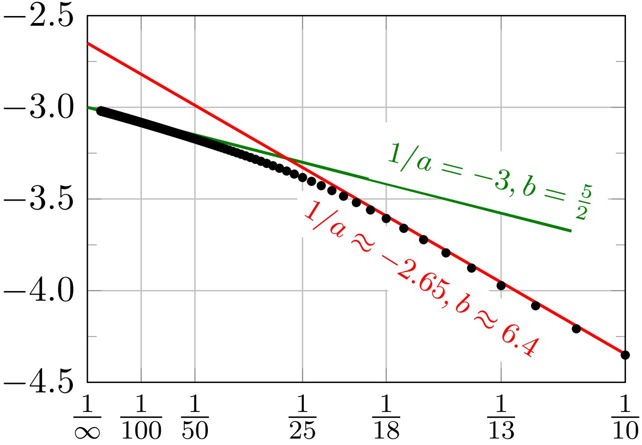
➡ instanton singularity is confluent

$$\mathfrak{B}(u) = \frac{A((u+\frac{1}{3})^{1/3}) + (u+\frac{1}{3})\log(u+\frac{1}{3}) \cdot B\Big((u+\frac{1}{3})\Big) + \dots}{(u+\frac{1}{3})^{7/2}}$$

with $A(w)$ and $B(w)$ analytic at $w = 0$

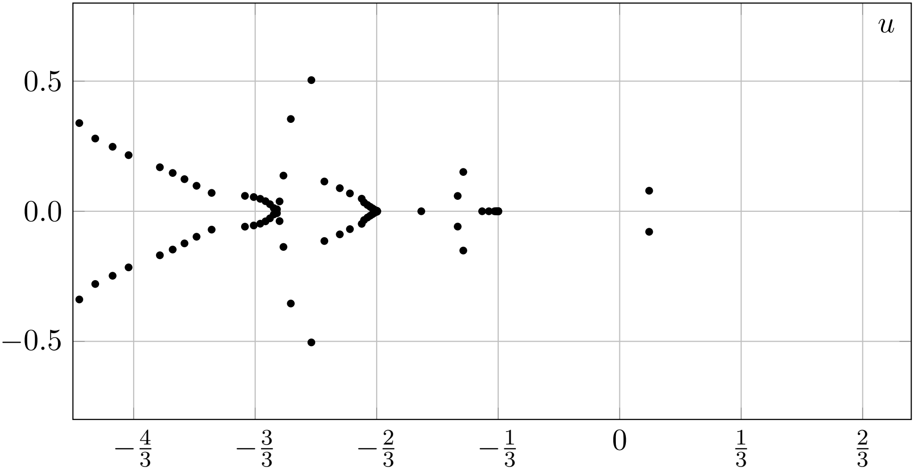
Ratio test:

$$\beta_n \sim C \cdot n! \cdot a^{-n} \cdot n^{b-1} \quad \Rightarrow \quad r_n := \frac{\beta_{n+1}}{n\beta_n} \sim \frac{1}{a} + \frac{b}{a}\frac{1}{n} + \mathcal{O}\left(n^{<-1}\right)$$



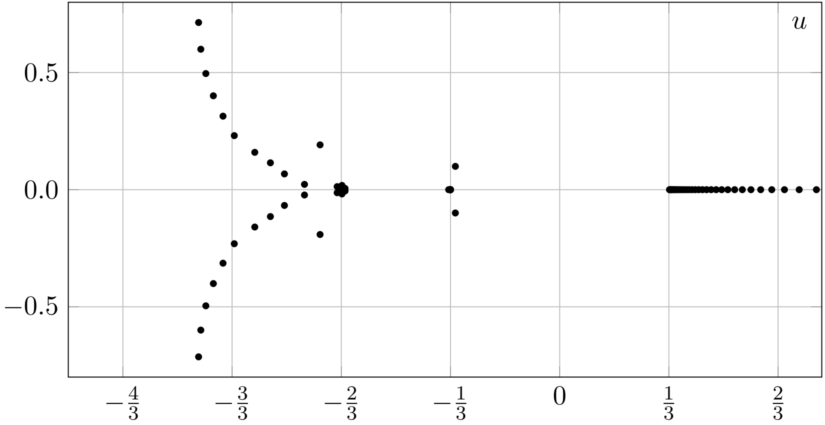
➡ approach of asymptotic regime is **very slow**!

poles of Padé approximant $\mathfrak{B}(u) = \frac{P(u)}{Q(u)}$, deg P = deg Q = 100:



- see instantons at $u = -1/3$, $u = -2/3$, $u = -3/3$, ...
- nothing else! (e.g. no renormalons)

Renormalization by subtraction at $p^2 = 0$:



- see instantons at $u = -1/3, -2/3, \ldots$ **but:**

$$\beta_n^{\mathrm{tr,MOM}} \sim C n! n^{3/2} (-3)^n \Big(1 + \ldots\Big) \sim \frac{\beta_n^{\mathrm{tr,MS}}}{n}$$

- renormalons at $u = 1/3$, $u = 2/3$, $\ldots$
- renormalon amplitude to 30 digits

Summary

- Very little is known about asymptotics in QFT.
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Summary

- Very little is known about asymptotics in QFT.
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- The tropical limit is an interesting new toy model of QFT.

  ➡ exact calculations to high order, correlations with actual QFT

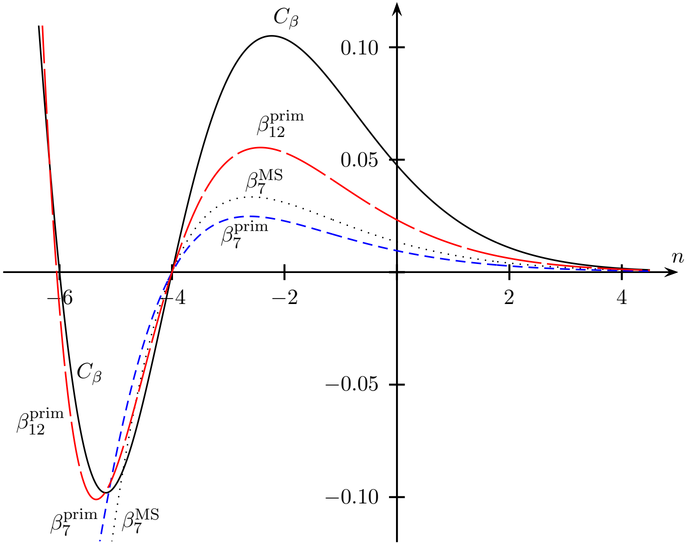
- Asymptotics are more complicated than

$$\beta_n \sim C \cdot n! \cdot n^{\alpha} \cdot \sigma^n \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \dots\right)$$

- tropical β-function is Borel summable in MS scheme
- renormalons do appear in kinematic schemes

Backup slides

$$\beta_k^{\mathrm{MS}} \sim k! \cdot k^{3+n/2} \cdot C(n) \qquad C(n) = \frac{36 \cdot 3^{(n+1)/2}}{\pi \Gamma(2+n/2) A^{2n+4}} e^{-3/2-(\gamma_E+3/4)(n+8)/3}$$



Consider scalar fields $\phi = (\phi_1, \ldots, \phi_n)$ with $O(n)$ symmetric interaction $\phi^4 := (\phi^2)^2$. The renormalized Lagrangian in $D = 4 - 2\varepsilon$ dimensions is

$$\mathscr{L} = \frac{1}{2} m^2 Z_1 \phi^2 + \frac{1}{2} Z_2 (\partial\phi)^2 + \frac{16\,\pi^2}{4!} Z_4\, g\, \mu^{2\varepsilon}\, \phi^4.$$

The Z-factors relate the renormalized (ϕ, m, g) to the bare (ϕ_0, m_0, g_0) via

$$Z_\phi = \frac{\phi_0}{\phi} = \sqrt{Z_2}, \quad Z_{m^2} = \frac{m_0^2}{m^2} = \frac{Z_1}{Z_2} \quad \text{and} \quad Z_g = \frac{g_0}{\mu^{2\varepsilon} g} = \frac{Z_4}{Z_2^2}.$$

Definition (RG functions: β and anomalous dimensions)

$$\beta(g) := \mu \frac{\partial g}{\partial \mu}\bigg|_{g_0} \quad \gamma_{m^2}(g) := -\mu \frac{\partial \log m^2}{\partial \mu}\bigg|_{m_0} \quad \gamma_\phi(g) := -\mu \frac{\partial \log \phi}{\partial \mu}\bigg|_{\phi_0}$$

$$\Gamma_R^{(k)}(\vec{p_1}, \ldots, \vec{p_k}; g, m, \mu) = Z_\phi^k \Gamma_0^{(k)}(\vec{p_1}, \ldots, \vec{p_k}; g_0, m_0, \mu)$$

Some $O(n)$ universality classes

$O(0)$ **self-avoiding walks**: diluted polymers

$O(1)$ **Ising model**: liquid-vapor transition, uniaxial magnets

$O(2)$ **XY universality class**: λ-transition of $^4\mathrm{He}$, plane magnets

$O(3)$ **Heisenberg universality class**: isotropic magnets

Onsager's solution from 1944

Exact critical exponents of the Ising model in $D = 2$ dimensions:

$$\alpha = 0, \quad \beta = 1/8, \quad \nu = 1, \quad \eta = 1/4.$$

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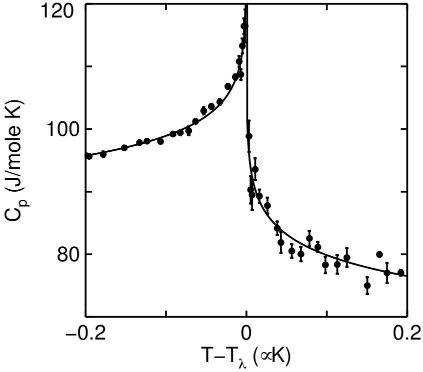
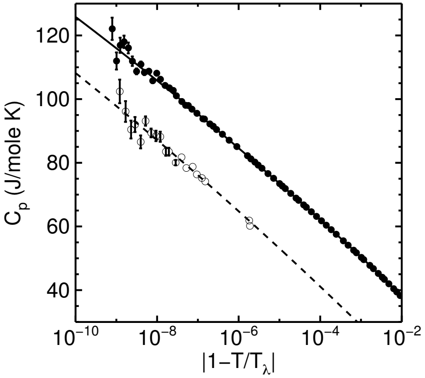
So far, no exact solutions in $D = 3$ are known. Approximation methods:

1. lattice: Monte Carlo simulation, high temperature series
2. conformal bootstrap (recently: very high accuracy for $n = 1$)
3. RG (ϕ^4 theory): in $D = 3$ dimensions
4. RG (ϕ^4 theory): in $D = 4 - 2\varepsilon$ dimensions (ε-expansion) $\Leftarrow$ this talk

λ-transition of ^{4}He (Columbia, October 1992)

Specific heat of liquid helium in zero gravity very near the lambda point [Lipa, Nissen, Stricker, Swanson & Chui '03]





Near the lambda transition ($T_\lambda \approx 2.2K$), the specific heat

$$C_p = A^{\pm} |t|^{-\alpha} \left(1 + a_c^{\pm} |t|^{\theta} + b_c^{\pm} |t|^{2\theta} + \cdots \right) + B^{\pm} \quad (\text{for } T \gtrless T_\lambda)$$

shows a power-law behaviour ($t = 1 - T/T_\lambda$). $\Rightarrow \alpha = -0.0127(3)$

RG equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-k\gamma_\phi-\gamma_{m^2}m^2\frac{\partial}{\partial m^2}\right]\Gamma_R^{(k)}\left(\vec{p_1},\ldots,\vec{p_k};m,g,\mu\right)=0$$

Near an IR-stable fixed point $g_\star$, that is

$$\beta(g_\star)=0 \quad \text{and} \quad \beta'(g_\star)>0,$$

the RG equation is solved by power laws and the critical exponents are

$$1/\nu=2+\gamma_{m^2}(g_\star), \quad \eta=2\gamma_\phi(g_\star) \quad \text{and} \quad \omega=\beta'(g_\star).$$

(scheme independent)

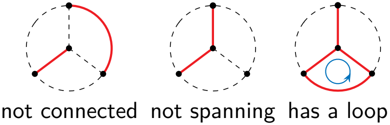
Recall specific heat near λ-transition of $^4\mathrm{He}$

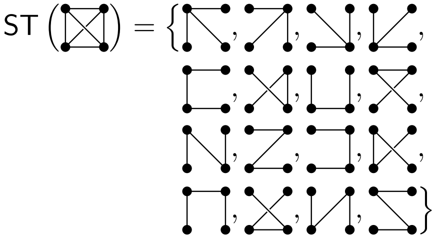
$$C_p=A^{\pm}\left|t\right|^{-\alpha}\left(1+a_c^{\pm}\left|t\right|^{\theta}+b_c^{\pm}\left|t\right|^{2\theta}+\cdots\right)+B^{\pm} \quad (\text{for } T\gtrless T_\lambda)$$

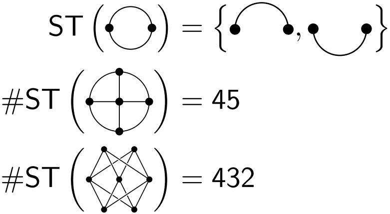
Here $\theta=\omega\nu\approx 0.529$ and $\alpha=2-D\nu\approx -0.0127$.

Spanning trees (ST)

A spanning tree $T \subset G$ is a simply connected, spanning subgraph.







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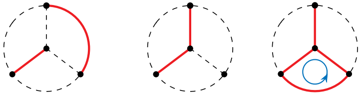

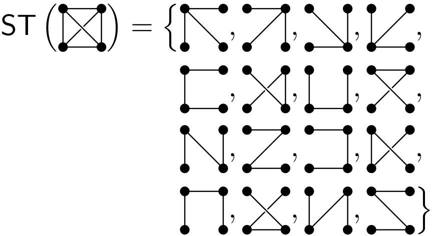

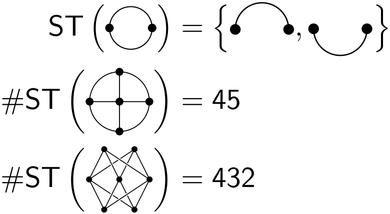

Symanzik polynomial

$$\mathcal{U}_G = \sum_{T \in \mathrm{ST}(G)} \prod_{e \notin T} x_e$$

Feynman period

$$\mathcal{P}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}_G^2 \big|_{x_1=1}}$$

$$\mathcal{U}_{\bigcirc} = x_1 + x_2$$

$$\mathcal{P}\left(\bigcirc\right) = \int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$$