UNIVERSE+ Online Seminar

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“Canonical functions and rational approximations”

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Canonical functions

& Rational approximations

Universe + seminar

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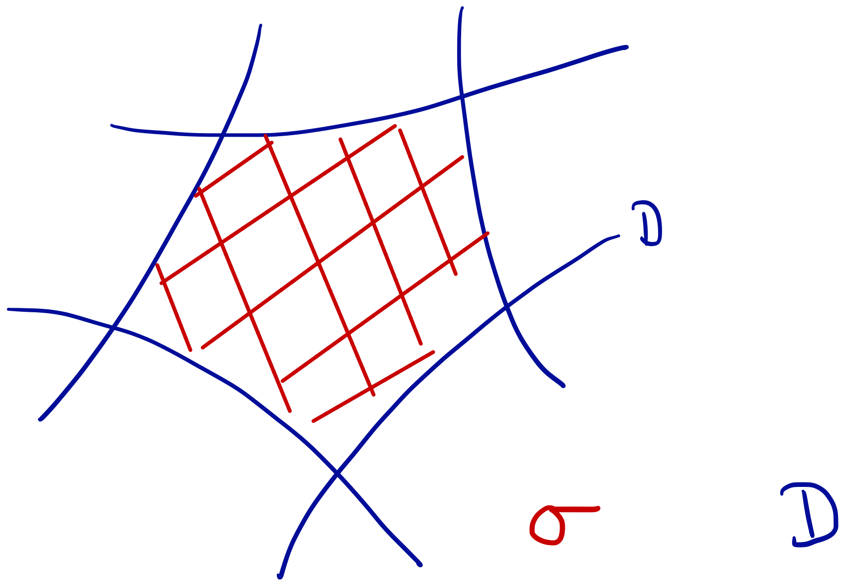
I/ Introduction

X smooth algebraic variety, dimension d.

$\sigma \subseteq X(\mathbb{C})$ semialgebraic region dim. d
"polytope"

$\partial\sigma \subseteq D(\mathbb{C})$ $D \subseteq X$ divisor



Want: Differential form $\omega_\sigma \in \Omega^d(X \setminus D)$ which has (logarithmic) poles along D.

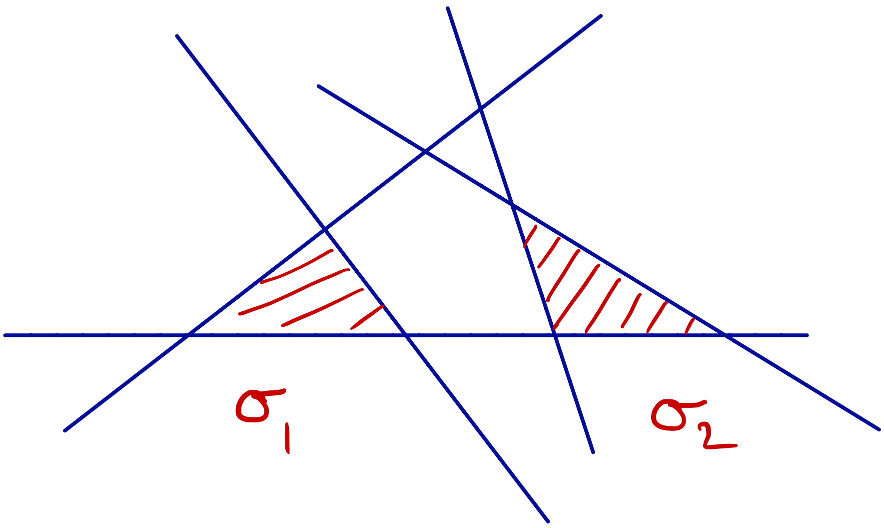
Why do we want to do this?

More interesting: take **two** semialgebraic sets σ_1, σ_2 which are "far away".

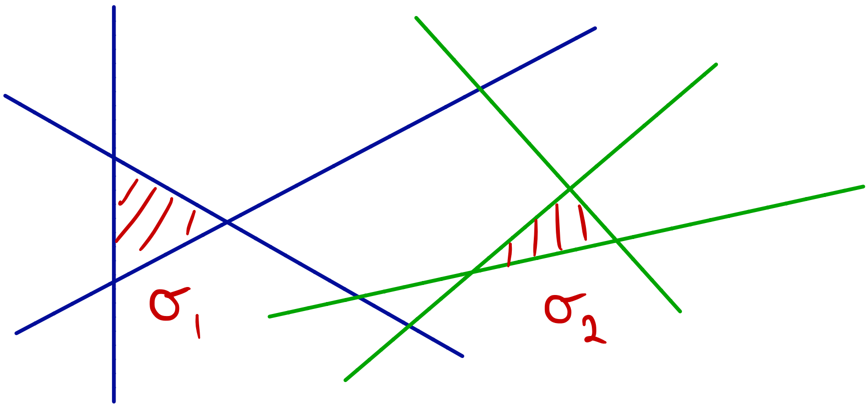
$$\left.\begin{aligned} \partial\sigma_1 &\subseteq D_1(\mathbb{C}) \\ \partial\sigma_2 &\subseteq D_2(\mathbb{C}) \end{aligned}\right\}$$

D_1, D_2 have no common components

"admissible".



BAD!



GOOD!

Suppose ω_{σ_2} has no poles on σ_1.

We will consider convergent integrals:

$$I = \int_{\sigma_1} \omega_{\sigma_2}$$

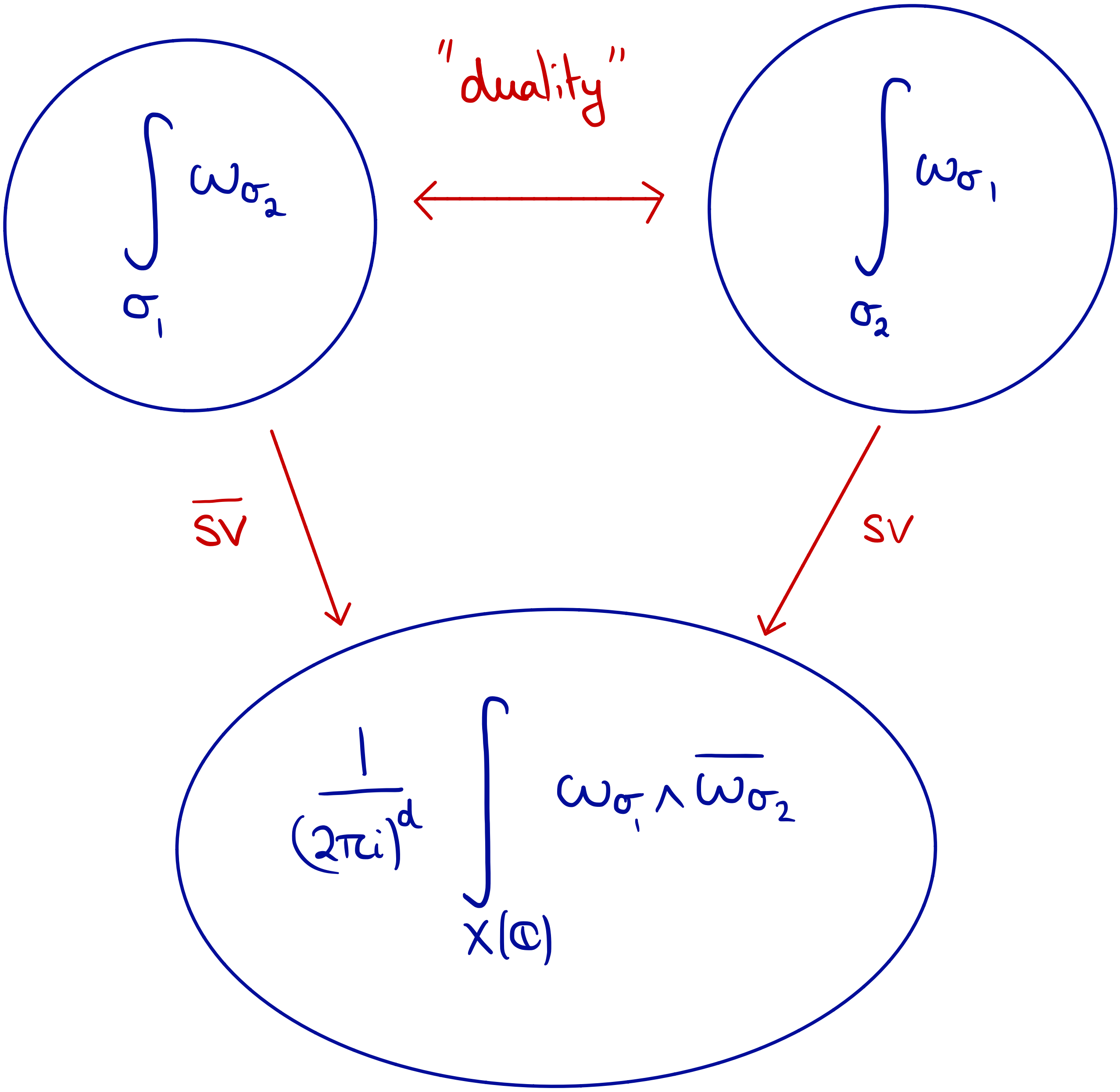
When D, X, ω_{σ_2} defined over $\mathbb{Q}$, these give very interesting periods.

But we can also switch the roles of σ_1, σ_2.

$$\int_{\sigma_2} \omega_{\sigma_1}$$

(Need orientations on σ_1, σ_2)

A triangle of periods.



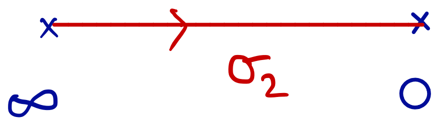
"duality" = Poincaré duality, "antipodal" duality

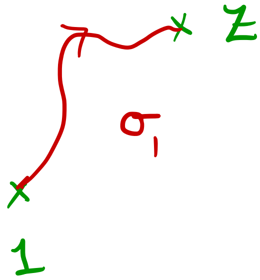
"sv" = single-valued map

double copy (w/ Dupont)

open versus closed string amplitudes.

Example : $X = \mathbb{C}$





$$\omega_{\sigma_2} = \frac{dx}{x} \qquad \omega_{\sigma_1} = d\log\left(\frac{x-z}{x-1}\right)$$

$$\log z = \int_1^z \frac{dx}{x} \quad \longleftrightarrow \quad -\log z = \int_\infty^0 \frac{dx}{x-z} - \frac{dx}{x-1}$$

$$\log |z|^2 = \frac{1}{2\pi i} \int_{\mathbb{C}} \left(\frac{1}{x-z} - \frac{1}{x-1}\right) \frac{1}{\bar{x}}\, dx\, d\bar{x}$$

II/. Canonical functions

In situations when ω_σ is unique and logarithmic it is called a canonical form.

Construction:

Suppose that σ_1, σ_2 admissible.

Let f_{σ_1/σ_2} be a rational function on X with

- simple poles along the boundary of σ_2
- simple zeroes along the boundary of σ_1

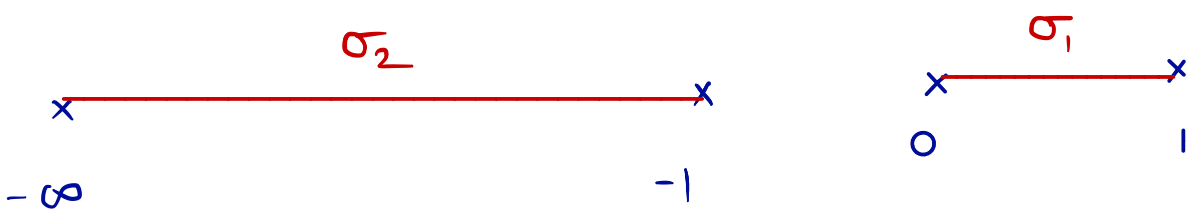
1412.6508 (2014)

If there is a unique f (up to scalar multiple) with

a $\left\{ \begin{matrix} \text{pole} \\ \text{zero} \end{matrix} \right\}$ of order 1 along each irreducible component of $\left\{ \begin{matrix} D_2 \\ D_1 \end{matrix} \right\}$

then it is (the) canonical function of (σ_1, σ_2).

Example : $\sigma_1 = [0,1]$ $\quad \sigma_2 = [-1, -\infty]$



$$\omega_{\sigma_2} = \frac{dx}{x+1}$$

$$f_{\sigma_1/\sigma_2} = \frac{x(1-x)}{x+1}$$

small on σ_1
big on σ_2.

Then f_{σ_1/σ_2} is a canonical function of the pair (σ_1, σ_2).

NB : $$f_{\sigma_2/\sigma_1} = \frac{x+1}{x(1-x)} = f_{\sigma_1/\sigma_2}^{-1}$$

<u>NB</u>: $f^n_{\sigma_1/\sigma_2} \, \omega_{\sigma_2}$

has poles of order n+1 along boundary of σ_2.

From now on we'll look at examples of families of integrals

$$I_{\sigma_1/\sigma_2}(n) = \int_{\sigma_1} f^n_{\sigma_1/\sigma_2} \, \omega_{\sigma_2} \quad , \quad n \geq 0$$

They give a very interesting infinite sequence of numbers $I(0), I(1), I(2), \ldots$

Example $\sigma_1 = [0,1]$ $\sigma_2 = [-\infty,-1]$

$$I(n) = \int_0^1 \left(\frac{x(1-x)}{1+x}\right)^n \frac{dx}{1+x}$$

$$I(0) = \log 2$$

$$I(1) = -2 + 3\log 2$$

$$I(2) = -9 + 13 \log 2$$

$$I(3) = \frac{-131}{3} + 63 \log 2$$

$$I(4) = -\frac{445}{2} + 321 \log 2$$

$$I(5) = -\frac{34997}{30} + 1683 \log 2$$

The coeff. of $\log 2$ is an integer sequence

$$1, 3, 13, 63, 321, 1683, \ldots$$

with amazing properties.

⑨

- $\frac{x(1-x)}{1+x}$ is <u>maximised</u> at $x = \sqrt{2} - 1$

- $\max_{x \in [0,1]} \frac{x(1-x)}{1+x} = 3 - 2\sqrt{2} \approx 0.171\ldots$

- So $I(n) \longrightarrow 0$ like $(0.171)^n$

  exponentially fast decay.

eg. $I(100) \sim 10^{-70}$

- $I(n) = a_n \log 2 - b_n$

  $a_n \in \mathbb{Z}, \quad b_n \in \mathbb{Q}$

$\Longrightarrow$ $\frac{b_n}{a_n}$ is a very good approximation to $\log 2$.

eg: $n = 5$

$$I(5) = -\frac{34997}{30} + 1683 \log 2$$

$$\frac{b_5}{a_5} = \frac{34497}{50490} \approx \log 2$$

to 7 digits.

III Holonomic sequences

Definition: A sequence of numbers $u_1, u_2, \ldots$ is holonomic or P-recursive if it satisfies a polynomial recurrence relation $\forall\, n \geq 0$

$$p_r(n)\, u_{n+r} + \cdots + p_1(n)\, u_{n+1} + p_0(n)\, u_n = 0$$

where $p_1, \ldots, p_r \in \mathbb{Z}[n]$ polynomials in n.

Theorem: Suppose σ_1, σ_2 admissible, f_{σ_1/σ_2} as above. Then

$$I_{\sigma_1/\sigma_2}(n) = \int_{\sigma_1} f^n_{\sigma_1/\sigma_2}\, \omega_{\sigma_2}$$

is holonomic.

Example: $\zeta(2)$

(Apéry, Beukers)

Let $X = \mathbb{C}^2$ $\sigma = \{0 \le t_1 \le t_2 \le 1\}$

$$f = \frac{t_1(t_1-t_2)(t_2-1)}{(t_1-1)t_2}, \quad \omega = \frac{dt_1 dt_2}{(t_1-1)t_2}$$

$$I_n = \int_\sigma f^n \omega = a_n \zeta(2) + b_n$$

$a_n \in \mathbb{Z}, b_n \in \mathbb{Q}$

a_n, b_n solutions to

$$(n+1)^2 u_{n+2} - (11n^2 + 11n + 3) u_{n+1} - n^2 u_n = 0$$

$a_0 = 1, a_1 = 3$
$b_0 = 0, b_1 = 5$ } Proves $\zeta(2) \notin \mathbb{Q}$!

Example : $\zeta(3)$ (Apéry, Beukers)

$X = \mathbb{C}^3$, $\sigma = \{ 0 \leq t_1 \leq t_2 \leq t_3 \leq 1 \}$

$$f = \frac{t_1(t_2-t_1)(t_3-t_2)(t_3-t_1)}{t_2(t_1-1)(t_2-1)t_3} \qquad \omega = \frac{dt_1 dt_2 dt_3}{t_2(t_1-1)(t_2-1)t_3}$$

$$I = \int_\sigma f^n \omega = a_n \zeta(3) + b_n$$

is the famous Apéry sequence:

$$(n+1)^3 u_{n+2} - (2n+1)(17n^2+17n+5) u_{n+1} + n^3 u_n = 0$$

$a_0 = 1$, $a_1 = 5$, ...

$b_0 = 0$, $b_1 = 6$, ...

Good enough to prove $\zeta(3)$ is irrational

Comments.

- It is not known how to prove

  $\zeta(5) \notin \mathbb{Q}$!!

- The previous two examples come from canonical constructions on $\mathcal{M}_{0,n}$

- The hope is to produce super-fast approximations to amplitudes.

  (w/. E-Panzer)

- The integers an have remarkable arithmetic properties:
  - super-congruences
  - connections to modular forms

  ⋮

IV Geometric interpretation

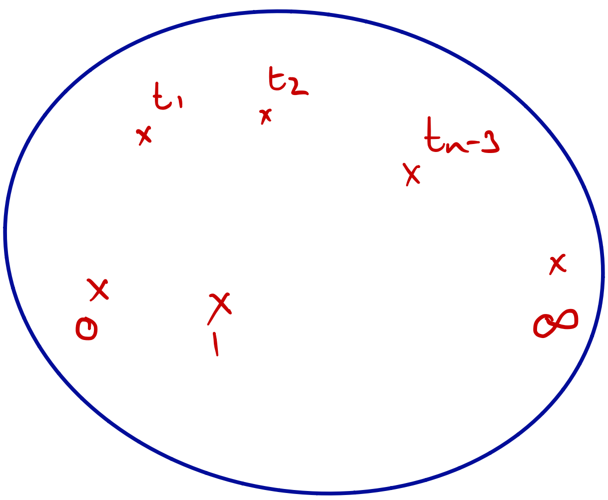
(i) Moduli space $\mathcal{M}_{0,n}$

$\mathcal{M}_{0,n}$ = moduli space of Riemann spheres with $n \geq 3$ ordered marked points

$$\mathcal{M}_{0,n}(\mathbb{C}) \cong \left\{ (z_1, \ldots, z_n) \in \mathbb{C}_\infty \right\} \Big/ PSL_2(\mathbb{C})$$

Use $PSL_2(\mathbb{C})$ - action to place $z_1 = 0,\ z_{n-1} = 1,\ z_n = \infty$

$$\cong \left\{ (0, t_1, \ldots, t_{n-3}, 1, \infty) \quad t_i \neq 0, 1, \infty, \quad t_i \text{ distinct} \right\}.$$



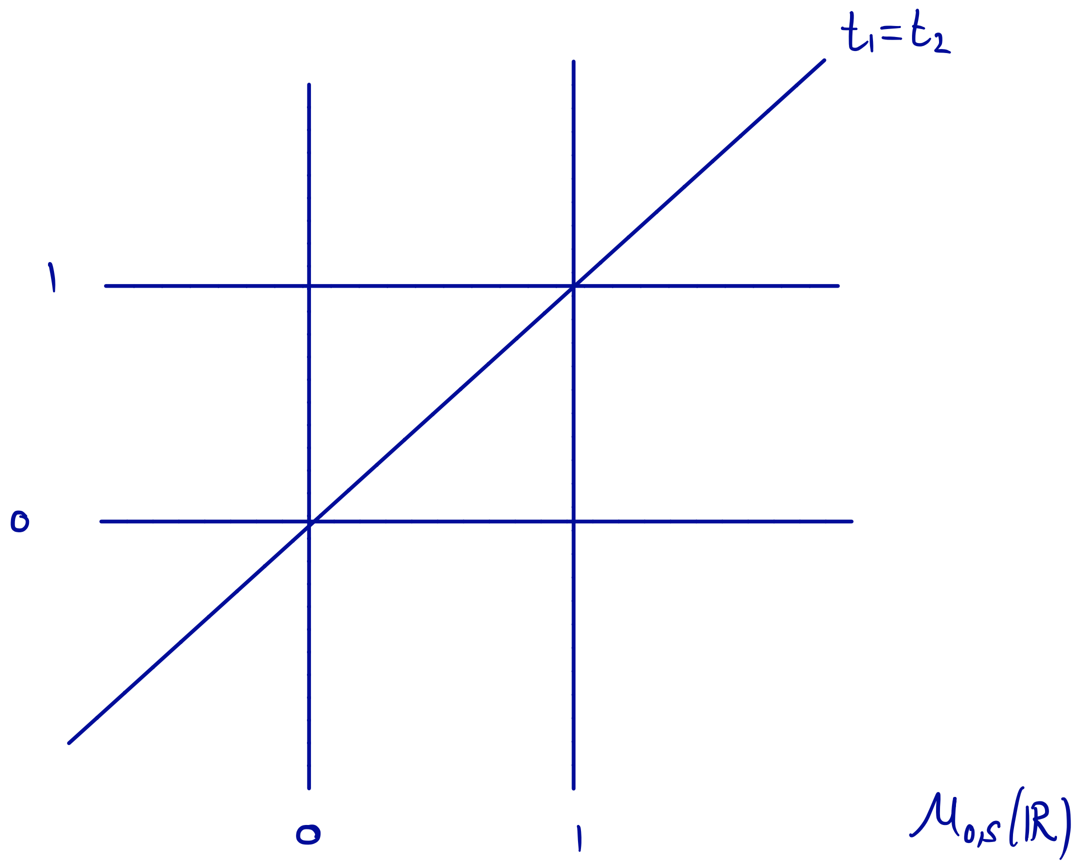
$n=4$:

$$\mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0,1\} \qquad t_1 \neq 0,1$$

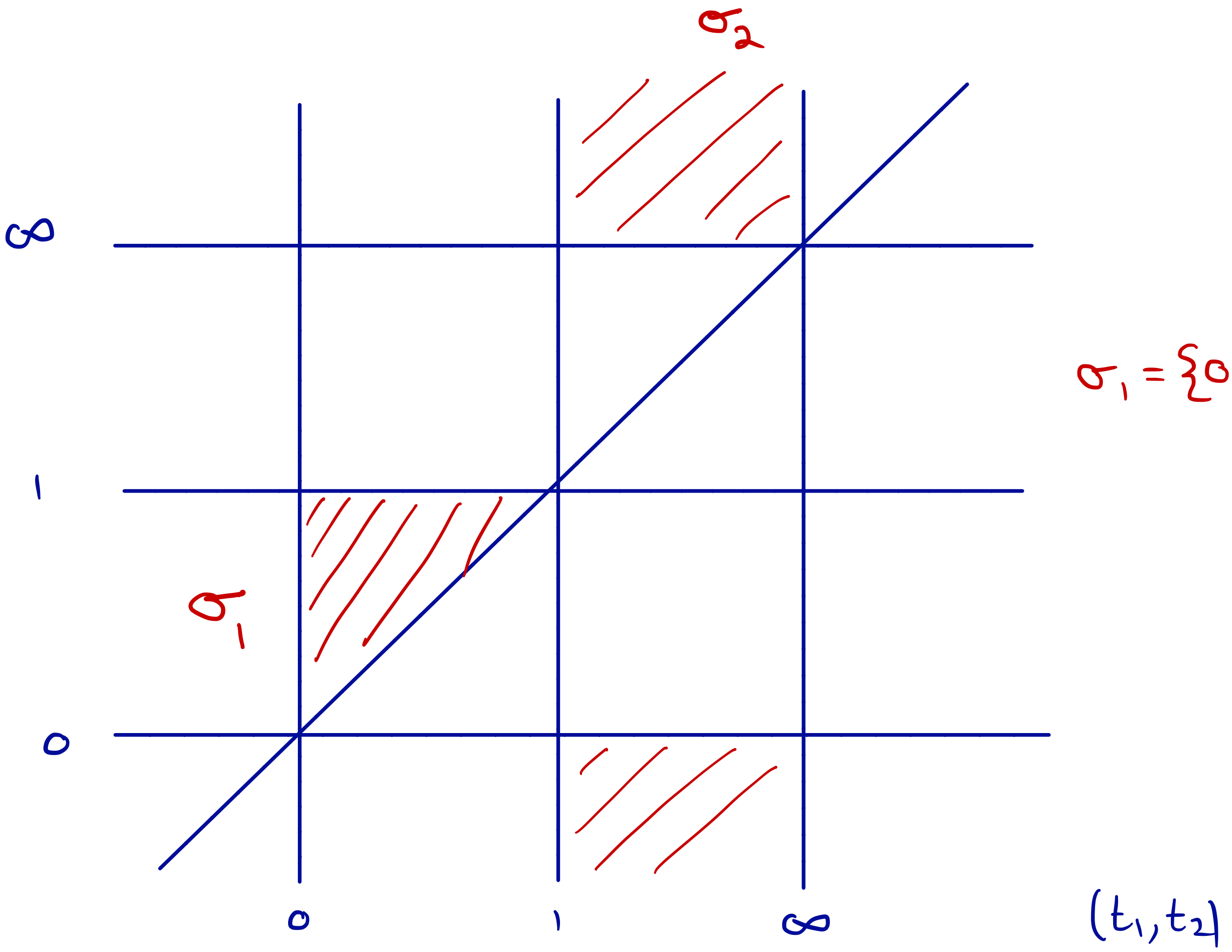
$n=5$:

$$\mathcal{M}_{0,5} \cong \mathbb{C}^2 \setminus \{ t_1=0,\ t_1=1,\ t_2=0,\ t_2=1,\ t_1=t_2 \}$$

(t_1, t_2)



The real points of $\mathcal{M}_{0,5}(\mathbb{R})$ has 12 connected components:



$\sigma_1 = \{0 \leq t_1 \leq t_2 \leq 1\}$

The regions σ_1, σ_2 are admissible.

Canonical form $\omega_{\sigma_2} = \dfrac{dt_1 dt_2}{(1-t_1)t_2}$

Canonical function $f_{\sigma_1/\sigma_2} = \dfrac{t_1(t_2-t_1)(1-t_2)}{(1-t_1)t_2}$

The integrals

$$I_{\sigma_1/\sigma_2}(n) = \int_{\sigma_1} f^n_{\sigma_1/\sigma_2} \, \omega_{\sigma_2}$$

are exactly the Apéry integrals for $\zeta(2)$.

$$I_{\sigma_1/\sigma_2}(0) = \int_{\sigma_1} \omega_{\sigma_2} = \zeta(2)$$

ii). Dihedral symmetry.

- The symmetric group Σ_n acts on $M_{0,n}$ by permuting the n marked points.

- The stabilizer of any connected component σ of $M_{0,n}(\mathbb{R})$ is $\cong$ to a dihedral group D_{2n}.

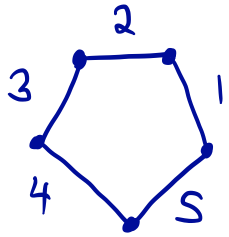
- Cells $\longleftrightarrow$ dihedral structures δ

Ex:

$$\{z_1 < z_2 < z_3 < z_4 < z_5 <\} \longleftrightarrow$$

$$\cong$$

$$\{0 < t_1 < t_2 < t_3 < 1\}$$



$(1\ 2\ 3\ 4\ 5)$

"standard cell"!

Theorem: (B. 2006, B.–Carr–Schneps 2009)

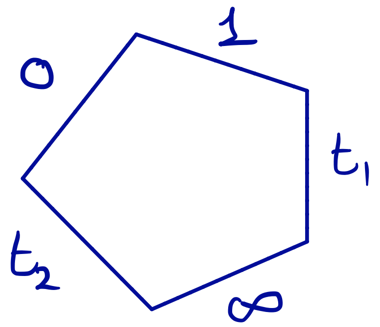
The canonical form associated to the cell σ_δ is

$$\omega_\delta = \pm \frac{dt_1 \wedge \dots \wedge dt_{n-3}}{\prod_{i \in \mathbb{Z}/n\mathbb{Z}} (z_{\delta_i} - z_{\delta_{i+1}})}$$

where : $z_1 = 0,\ z_2 = t_1,\ \dots,\ z_{n-2} = t_{n-3},\ z_{n-1} = 1$

& we omit all factors containing $z_n = \infty$.

Example : $\delta = (1\ 3\ 5\ 2\ 4)$



$$\omega_\delta = \pm \frac{dt_1\, dt_2}{(1-t_1)\, t_2}$$

Let (δ_0, δ) be two dihedral structures

assume : δ_0 = standard

$\sigma_{\delta_0} = \{0 \le t_1 \le \dots \le t_{n-3} \le 1\}$

$$f_{\delta_0/\delta} = \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \left(\frac{z_{\delta_0(i)} - z_{\delta_0(i+1)}}{z_{\delta(i)} - z_{\delta(i+1)}} \right)$$

canonical function

Eg:

$\delta_0 = (1\ 2\ 3\ 4\ 5)$

$\delta_0 = (1\ 3\ 5\ 2\ 4)$

$$f_{\delta_0/\delta} = \frac{t_1 (t_2 - t_1)(1 - t_2)}{t_2 (1 - t_1)}$$

iii) Cellular integrals

Let (δ_0, δ) two dihedral structures.

$$I_{\delta/\delta_0}(n) = \int_{\sigma_{\delta_0}} f^n_{\delta_0/\delta} \, \omega_\delta$$

$f^n_{\delta_0/\delta}$ ↑ canonical function

ω_δ ↑ canonical form

Defn: The **dual** cellular integral is

$$I_{\delta_0/\delta}(n) = \int_{\sigma_\delta} f^n_{\delta/\delta_0} \, \omega_{\delta_0}$$

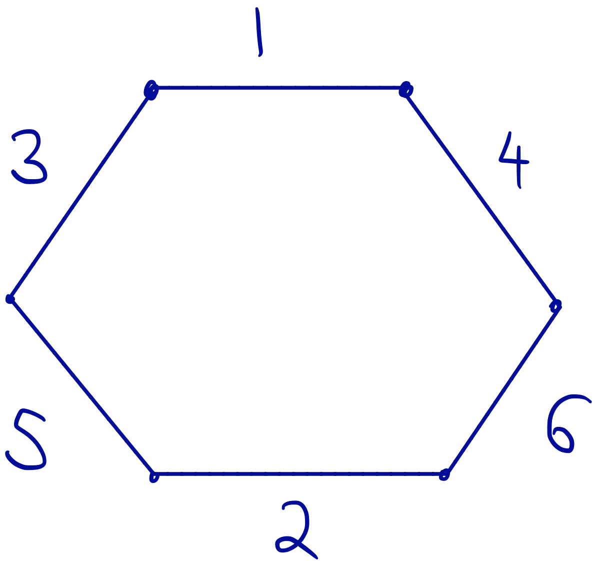
$I_{\delta/\delta_0}(n)$ does not always converge!

Convergence.

There are necessary & sufficient conditions on σ for the pair (σ_0, σ) to be admissible and for the integral to converge.

Roughly, σ should not have sequences of consecutive numbers modulo n.

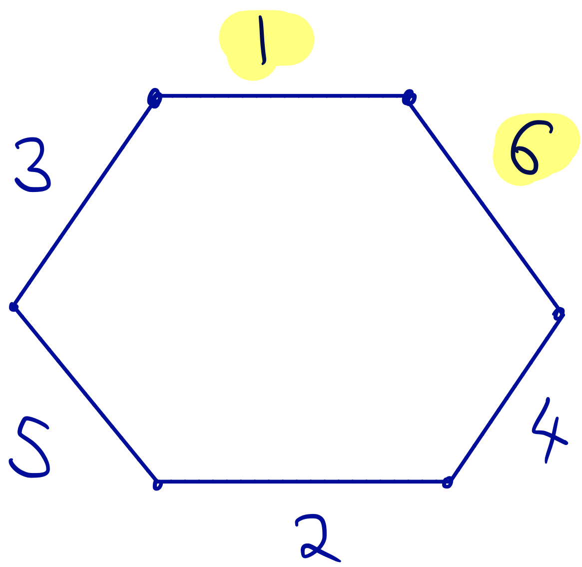
eg: $n = 6$



ok!

Dinner table problem!

(Poulet 1919)



Bad!

Up to symmetries, here are the number of admissible pairs (σ_0, σ) for each n:

n	4	5	6	7	8	9
# pairs $/D_{2n}$	0	1	1	5	17	105

For $n=5$, $n=6$ they are <u>unique</u>.

- $n=5$ we get Apéry's sequence for $\zeta(2)$
- $n=6$ we get Apéry's sequence for $\zeta(3)$.

$$(\sigma_0, \sigma) = \big((1\,2\,3\,4\,5\,6), (1\,3\,5\,2\,6\,4)\big)$$

- $n=7$ we get integrals which are linear combinations of $1, \zeta(2), \zeta(2)^2$.

For higher n, we get very interesting families but which do not (yet) lead to irrationality of ζ(5)! (28)

10.1.4. $N = 8$. There are 17 convergent configurations, comprising 7 pairs of configurations and their duals:

$$
\begin{aligned}
{}_8\pi_1 &= [8,2,4,1,5,7,3,6] \quad , \quad {}_8\pi_1^\vee = [8,2,5,1,7,4,6,3] \\
{}_8\pi_4 &= [8,2,4,7,1,6,3,5] \quad , \quad {}_8\pi_4^\vee = [8,2,4,7,3,6,1,5] \\
{}_8\pi_5 &= [8,2,5,3,7,1,6,4] \quad , \quad {}_8\pi_5^\vee = [8,2,6,1,5,3,7,4] \\
{}_8\pi_7 &= [8,2,4,6,1,3,7,5] \quad , \quad {}_8\pi_7^\vee = [8,2,5,1,6,3,7,4] \\
{}_8\pi_8 &= [8,2,5,1,6,4,7,3] \quad , \quad {}_8\pi_8^\vee = [8,2,4,1,7,5,3,6] \\
{}_8\pi_9 &= [8,2,5,7,3,1,6,4] \quad , \quad {}_8\pi_9^\vee = [8,3,6,1,5,2,7,4] \\
{}_8\pi_{10} &= [8,2,5,7,3,6,1,4] \quad , \quad {}_8\pi_{10}^\vee = [8,2,5,7,4,1,6,3]
\end{aligned}
$$

and three self-dual configurations:

$$
\begin{aligned}
{}_8\pi_2 = {}_8\pi_2^\vee &= [8,2,4,1,6,3,7,5] \\
{}_8\pi_3 = {}_8\pi_3^\vee &= [8,2,5,1,7,3,6,4] \\
{}_8\pi_6 = {}_8\pi_6^\vee &= [8,3,6,1,4,7,2,5]
\end{aligned}
$$

Configurations	1	$\zeta(2)$	$\zeta(3)$	$\zeta(4)$	$\zeta(5)$	$\zeta(3)\zeta(2)$	$I_\pi(0)$
${}_8\pi_1$, ${}_8\pi_1^\vee$	•	•	•	0	0	•	$2\zeta(2)\zeta(3)$
${}_8\pi_2$, ${}_8\pi_3^\vee$	•	•	•	0	•	•	$\zeta(5)+\zeta(3)\zeta(2)$
${}_8\pi_4$, ${}_8\pi_5$	•	•	•	0	•	•	$9\zeta(5)-2\zeta(2)\zeta(3)$
${}_8\pi_4^\vee$, ${}_8\pi_5^\vee$	•	•	•	0	•	•	$9\zeta(5)-4\zeta(3)\zeta(2)$
${}_8\pi_6$	•	•	•	0	•	•	$16\zeta(5)-8\zeta(3)\zeta(2)$
${}_8\pi_7$	•	•	•	0	•	•	$\zeta(5)+3\zeta(3)\zeta(2)$
${}_8\pi_7^\vee$	•	•	•	0	•	•	$\zeta(3)\zeta(2)-\zeta(5)$
${}_8\pi_8$	•	0	•	0	•	0	$2\zeta(5)$
${}_8\pi_8^\vee$	•	•	0	0	•	•	$2\zeta(5)+4\zeta(3)\zeta(2)$
${}_8\pi_9$	•	•	•	0	•	•	$6\zeta(3)\zeta(2)-7\zeta(5)$
${}_8\pi_9^\vee$	•	•	•	0	•	•	$4\zeta(3)\zeta(2)-7\zeta(5)$
${}_8\pi_{10}$	•	•	•	0	•	•	$5\zeta(3)\zeta(2)-8\zeta(5)$
${}_8\pi_{10}^\vee$	•	•	•	0	•	•	$8\zeta(5)-3\zeta(3)\zeta(2)$

Conclusion

- If a positive geometry has ≥ 2 canonical forms associated to admissible σ_1, σ_2 we get numbers

$$\int_{\sigma_1} \omega_{\sigma_2} \quad , \quad \int_{\sigma_2} \omega_{\sigma_1} .$$

- A canonical function gives a family of numbers

$$I_{\sigma_1/\sigma_2}(n) = \int_{\sigma_1} f^{n}_{\sigma_1/\sigma_2}\, \omega_{\sigma_2}$$

obtain a very rich mathematical structure:

- Holonomic recurrence relation
- Integer sequence with remarkable p-adic properties
- Surprising connections with modular forms
- Exotic birational symmetry groups.
- Multiplicative structures

⋮