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UNIVERSE+ Online Seminar

Mikhail Kapranov

“Landau singlarities and convex geometry”

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Landau singularities and convex geometry

based on [2609.27508]

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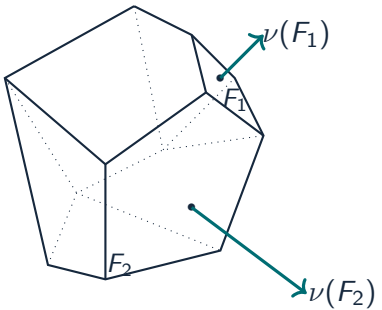
Minskowski problem: encoding "conserved momenta" by a polytope

$Q \subset \mathbb{R}^n$: full-dimensional convex polytope.
Each facet $F \rightsquigarrow$ the vector

$$\nu_Q(F) = |F|\, n_{Q,F}, \quad \text{where}$$

$|F|$ is its $(n-1)$-area, $n_{Q,F}$ its outward unit normal. We have:

$$(!) \qquad \sum_{F \text{ facet of } Q} \nu_Q(F) = 0.$$



In scattering theory:

Momenta p_i of incoming and outgoing particles satisfy $\sum p_i = 0$ (mom. conservation).
Can this way of producing such "conserved" sets $\{p_i\} = \{p_F = \nu_Q(F)\}$ play a role?

Proof of (!); facet weights without a Euclidean metric

E: just an oriented n-dimensional $\mathbb{R}$-space;
Ω: the constant $\Lambda^{n-1}E$-valued $(n-1)$-form given by the identity. Then:

$$\nu_Q(F) = \int_F \Omega \ \in \ \Lambda^{n-1}E \quad \text{invariantly, and}$$

$$\sum_{F \subset \partial Q} \nu_Q(F) = \int_Q d\Omega = 0 \ \in \ \Lambda^{n-1}E \quad \text{by Stokes.}$$

A choice of a volume form identifies $\Lambda^{n-1}E$ with E^*. A compatible Euclidean metric identified it with E and we get the usual definition.

Minkowski's inverse problem

Given a "conserved" set of vectors, can we make them the weighted normals of a polytope?

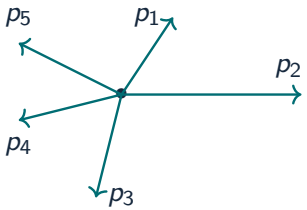
Minkowski's theorem: from external momenta into a polytope

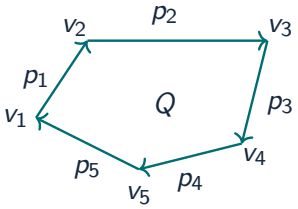
Let $p_1, \ldots, p_N \in \mathbb{R}^n$ be nonzero and span $\mathbb{R}^n$. Assume that they have distinct positive rays and that $\sum_i p_i = 0$.

Minkowski's theorem (1897)

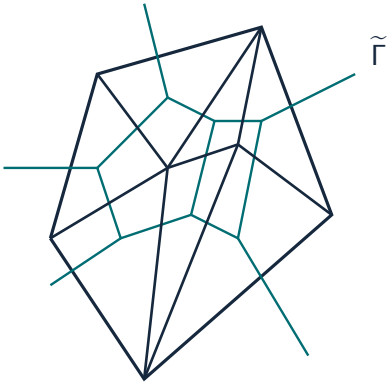
There is a full-dimensional polytope $Q(p)$, unique up to translation, with one facet for each p_i and $\nu_{Q(p)}(F_i) = p_i$.

In 2 dimensions the reconstruction is a polygon: Put the vectors in cyclic order and concatenate them, rotate by 90°.





A polyhedral subdivision $\mathcal{P}$ of Q gives a "scattering diagram"



$\widetilde{\Gamma} = \widetilde{\Gamma}_{\mathcal{P}}$: dual graph

- Cell $P \longleftrightarrow$ vertex of $\widetilde{\Gamma}$.
- Shared facet $F \longleftrightarrow$ internal edge.
- Boundary facet $\longleftrightarrow$ external leg.

On a shared facet $F = P \cap P'$ we have

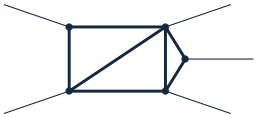
$$\nu_P(F) = -\nu_{P'}(F).$$

At each cell, "mom. conservation".

$\Rightarrow$ we get a scattering diagram, i.e., a 1-cycle $\nu \in C_1(\widetilde{\Gamma}, V) = \bigoplus_{e \in \mathrm{Ed}(\widetilde{\Gamma})} \mathrm{or}(e) \otimes V$, $V = \mathbb{C}^n = \{\mathbb{C} - \text{momenta}\}$. (Cellular chain complex of the graph with coefficients in V)

Feynman integral (over loop momenta) associated to a graph $\widetilde{\Gamma}$

Remove the legs:



$\rightsquigarrow$



$\mathrm{Ed}(\widetilde{\Gamma}) = L \sqcup \mathrm{Ed}(\Gamma)$

$\widetilde{\Gamma}$ $\qquad$ Γ

Boundary: $\partial : C_1(\Gamma, V) \to C_0(\Gamma, V) \ni p = (p_l)_{l \in L}$: datum of external momenta.

$$I(p) = \int_{k \in \gamma(p) \subset \partial^{-1}(p)} \frac{B(p,k)dk}{\prod_e A_e(k_e)}, \quad \frac{1}{A_e(k_e)} = \frac{1}{m_e^2 - k_e^2} = \text{propagator}$$

$B(p,k)$: some polynomial (details of interaction, does not affect sing. behavior).
$\gamma(p)$: cycle avoiding singularities & varying by GM connection.
m_e = mass of particle "associated to edge e". Where is $I(p)$ singular?

Landau multipliers

NB: Integration cycle $\gamma(p)$ lies in $\partial^{-1}(p) - \bigcup$ quadric hypersurf.$\{A_e(k_e) = 0\}$

Cayley/Feynman/Schwinger trick

(1) Degenerate behavior of arrangement of hupersurf. $\{f_i(x) = 0\} \Leftrightarrow$ degen. behavior of a single hypersurf. $\{F(\alpha, x) = \sum \alpha_i f_i(x) = 0\}$ w.r.t. coord. cross $\{\alpha_i = 0\}$.

(2)
$$\int\limits_{x\in\gamma} \frac{dx}{f_1(x)^{s_1}\cdots f_n(x)^{s_n}} = \frac{\Gamma(\sum s_i)}{\prod \Gamma(s_i)} \int\limits_{x\in\gamma,\alpha\in\Delta^{n-1}} \frac{\alpha_1^{s_1-1}\cdots\alpha_n^{s_n-1}}{F(\alpha,x)^{s_1+\cdots+s_n}}\, d\alpha.$$

Δ^n: simplex $\{\alpha \in \mathbb{R}^n | \alpha_{\geq} 0, \sum \alpha_i = 1\}$.
(3) Version of (2) with exponentials and integration over all $\alpha \in \mathbb{R}^n$

Landau (1959) Equations 1: For some branch of $I(p)$ to be singular at $p^{(0)}$,
$F(\alpha, k) = \sum \alpha_e A_e(k_e)$ on $(\mathbb{C}^*)^{\mathrm{Ed}(\Gamma)} \times \partial^{-1}(p)$ must have a crit. pt. at some $(\alpha^{(0)}, q)$.

Landau equations over $\mathbb{C}$: coboundary/spatial/Hodge theory formulation

Tangent space $T_q\partial^{-1}(p) \simeq \partial^{-1}(0)$, cycle space. Differential is $d_q(A_e) = -2(q_e, dq)$. $\Rightarrow$

Landau Eqs. 2: (α, q) critical for $F \Rightarrow$ Tuple $(\alpha_e q_e)_{e\in \mathrm{Ed}(\Gamma)} \in C_1(\Gamma, V)$ is orthogonal to $\partial^{-1}(0)$, i.e.: Considered as a cochain ($C_1 \simeq C^1$ as $V \simeq V^*$), it is a coboundary $\delta\varphi$.

$$C^0(\Gamma, V) \underset{\partial}{\overset{\delta=\partial^*}{\rightleftarrows}} C^1(\Gamma, V).$$

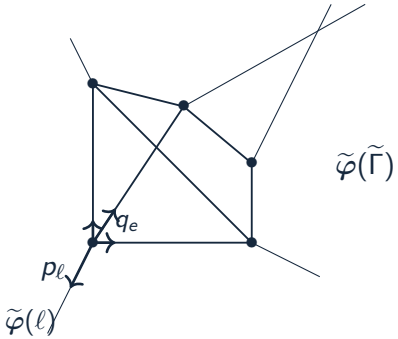
So $\boxed{\alpha_e q_e = \varphi(w) - \varphi(v)}$, if $e = [v, w]$. Spatial realization (immersion) $\varphi : \Gamma \to V$.

Landau Eqs. 3:

p a singularity if $p = \partial(\alpha^{-1} \cdot \delta(\varphi))$ for some φ, where $\alpha^{-1} = (\alpha_e^{-1})$ stretch $\delta\varphi$ so as $\|\alpha_e^{-1} \cdot (\delta\varphi_e)\|^2 = m_e^2$. Critical point is (α, q), $q = \alpha^{-1} \cdot \delta\varphi$.

NB: $(\alpha_e) \sim$ metric on spatially immersed Γ, and $\mathbb{D} = \partial \circ \alpha^{-1} \circ \delta \sim$ graph Laplacian.

Landau graph immersed in space



Extend $\varphi : \Gamma \to V$ to $\widetilde{\varphi} : \widetilde{\Gamma} \to V$ by adding for each leg ℓ a half-line in direction of $p_\ell = (\mathbb{D}\varphi)_\ell$. The vector q_e is proportional (via α_e^{-1}) to the direction of the edge e.

NB. Images of edges and legs can cross ($\widetilde{\varphi}$ need not be an embedding).

Real aspects: positivity

Most immediately interesting:

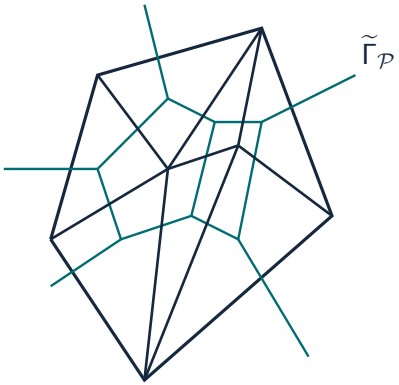
- Momenta p_ℓ real, so now $V = \mathbb{R}^n$ and Γ immersed into $\mathbb{R}^n$.
- $\alpha_e > 0$ real positive (if some $\alpha_e = 0$, the edge is contracted, Γ gets smaller)

In other words, we look for positive solutions, in which:

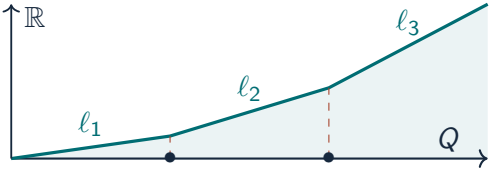
The edge vector $\varphi(w) - \varphi(v)$ for the immersed graph points along the momentum q_e for the chosen orientation. The multiplier α_e is the positive scale between them.

(Like in the previous picture)

From convex data to a positive solution



Now $\widetilde{\Gamma} = \widetilde{\Gamma}_{\mathcal{P}}$ for a subdivision $\mathcal{P}$ of Q. Suppose $\mathcal{P}$ is regular, i.e., there is a continuous convex piecewise-affine function $f : Q \to \mathbb{R}$ whose maximal affine domains are the full-dimensional cells of $\mathcal{P}$.



Thm. Such f gives a positive solution of Landau's equations with $m_e = |F_e|$ being the area of the face P_e corresponding to the edge e (in Euclidean signature).

Proof: Continuity gives the direction, convexity gives the sign

Take adjacent cells P, P' and orient $e : P \to P'$ across their common facet F. Set $\varphi(P) = \ell_P = d(f|_P)$. This embeds the dual graph into the slope space.

$$(\ell_{P'} - \ell_P)|_{TF} = 0 \qquad \text{(the two affine pieces agree on } F\text{)},$$
$$(\ell_{P'} - \ell_P)(u) > 0 \qquad (u \text{ points from } P \text{ to } P' \text{ across } F).$$

Therefore

$$\ell_{P'} - \ell_P = \alpha_e \, \nu_P(F), \qquad \alpha_e > 0.$$

Here we use the Euclidean pairing to identify normals and covectors. $\Rightarrow$ Landau's Eqs.:

$$\underbrace{\sum_{F \subset P} \nu_P(F) = 0}_{\text{conservation}} \qquad \underbrace{q_e^2 = m_e^2}_{\text{mass shell}} \qquad \underbrace{\alpha_e q_e = \varphi(P') - \varphi(P)}_{\text{Landau proportionality}}.$$

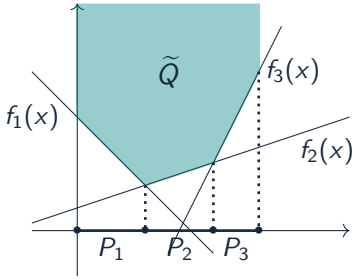
Remarks

(1) In Lorentzian signature $(+,-,\ldots,-)$, a spacelike normal cannot give a real mass. The positive Landau solutions can still be obtained from subdivisions with nonspacelike normals.

(2) A fixed physical theory prescribes masses in advance. In this case, finding a subdivision with the required facet areas is an extra constraint. The theorem can also be used in the reverse direction to generate examples with masses read from the geometry.

(3) Not every positive Landau solution can be obtained from a subdivision. E.g., in n dimensions the dual graphs have all vertices of valence $\geq n+1$.

The Aleksandrov problem



Given: a polytope $Q \subset \mathbb{R}^n$ and *linear* functions $\ell_1, \cdots \ell_m : \mathbb{R}^n \to \mathbb{R}$. Vary constant terms: put $f_i(x, h) := \ell_i(x) + h_i$ and define

$$f(x, h) = \max_i f_i(x, h)$$

convex, defines a regular subdivision $\mathcal{P}(h)$ into $P_i(h) = \{f = f_i\}$. Put $V_i(h) = \operatorname{Vol} P_i(h)$ (may be 0)

Pb.: Given $V_i > 0$ summing up to $\operatorname{Vol}(Q)$, find h_i s.t. $V_i(h) = V_i$.

(Version of Minkowski Pb. but for unbounded polyhedron $\widetilde{Q}$)

Aleksandrov's Thm (1951): Always possible uniquely up to common add. constant.

Variational formula for Landau multipliers

Aleksandrov problem tells us to:

$$\text{Study the map } h = (h_1, \cdots, h_m) \bmod \mathbb{R} \mapsto (V_1(h), \cdots, V_m(h)),$$
$$\text{prove that it's } 1:1 \text{ over interior of } \Delta^{m-1}.$$

Prop.(Gu, Luo, Sun, Yau [1302.5472]):

$$\frac{\partial V_i}{\partial h_j} = \frac{A_{ij}}{\|\ell_i - \ell_j\|}, \quad \text{where}$$

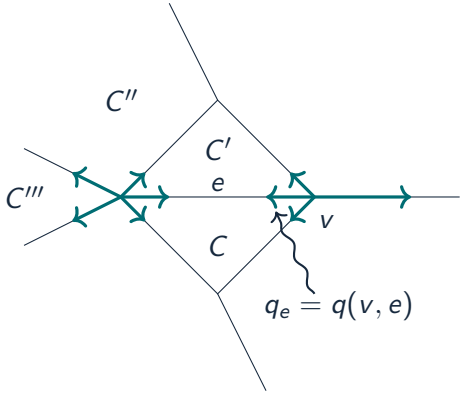
A_{ij} = area of face F_{ij} between P_i and P_j (zero if no such face).

Note:

- Relevant $(i,j) \leftrightarrow$ edges e of dual graph $\widetilde{\Gamma}$.
- RHS = Landau multiplier, i.e.:

Corollary: If $e = [v_i, v_j]$ joins vertices corresponding to P_i and P_j, then $\alpha_e = \partial V_i / \partial h_j$.

Planar case: reconstruction (Landau 1959 method of schemes formalized)



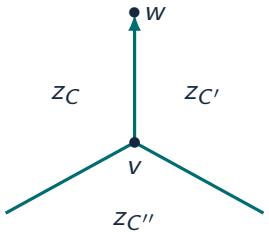
Prop. Any positive solution of Landau equations s.t.:

- $\widetilde{\varphi} : \widetilde{\Gamma} \hookrightarrow \mathbb{R}^2$ is an embedding, i.e., straight edges and half-infinite rays do not cross.
- Boundary data are nondegenerate: no parallel rays

is polyhedral/polygonal: it comes from a regular subdivision of the Minkowski polygon of external momenta.

Pf: Use regions $C \subset \mathbb{R}^2 - \widetilde{\Gamma}$. Find points x_C that jump along each e by q_e. Possible by conservation around each v. The x_C for unbounded cells give Q. The x_C for C around v give P_v (all after rotation by 90°). The multipliers α_e reconstruct $f : Q \to \mathbb{R}$.

Gaiotto–Moore–Witten [1506.04086] webs = planar Landau graphs



Their motivation: 2dim. $N = (2, 2)$ SUSY theory, scattering of solitons. Data (a *plane web*):

- $\Gamma \subset \mathbb{C}$ embedded graph just as before
- Regions C labelled by $z_C \in \mathbb{C}$ (cent. charges) so that:
- For a finite edge from v to w we have

$$w - v = \alpha_e(z_C - z_{C'}), \qquad \alpha_e > 0.$$

This is the Landau condition.

Different physics, same mathematics!

Soliton/instanton structures in GMW vs. propagator pinches (change of topology of integration space) in Landau.

The dictionary between plane webs and polygonal subdivisions

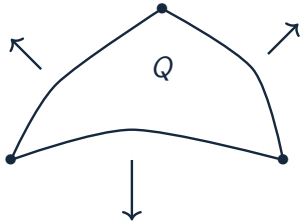
Plane web	**Convex geometry and Landau data**
Region label z_C	Dual polygon vertex $x_C = \mathrm{Rot}_{90^\circ}^{-1}(z_C)$
Edge separating C, C'	Side between $x_C, x_{C'}$ with momentum $z_C - z_{C'}$
Web vertex v	Polygonal cell with slope ℓ_v
Finite edge vector $w - v$	Positive multiplier times the internal momentum
External rays	Boundary sides of the momentum polygon

K., Kontsevich, Soibelman, *Algebra of the Infrared and secondary polytopes*, §13.

Final remarks/thoughts

(1) Role or place of polyhedral Landau solutions in higher dimensions?

(2) Curvilinear polyhedra Q: the "averaged normals" $\nu_Q(F) = \int_F \Omega$ still sum up to 0. Can one use these?



(3) Is there a version of the Minkowski problem for "matrix/Grassmannian convexity theory" (with totally positive matrices instead of $\mathbb{R}_+$)?