# KINEMATIC FLOW

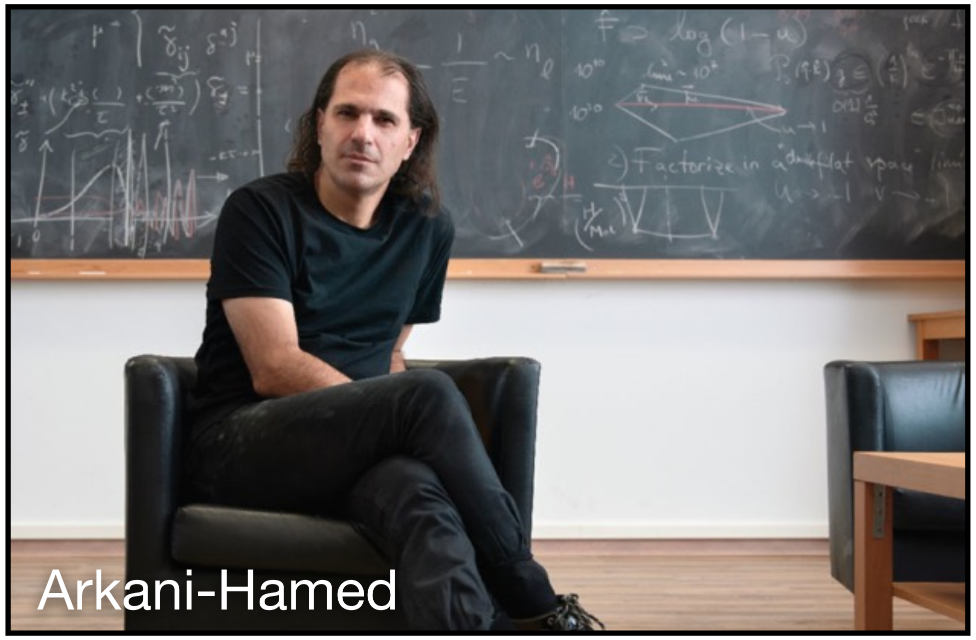
## A Hidden Pattern in Cosmological Correlations

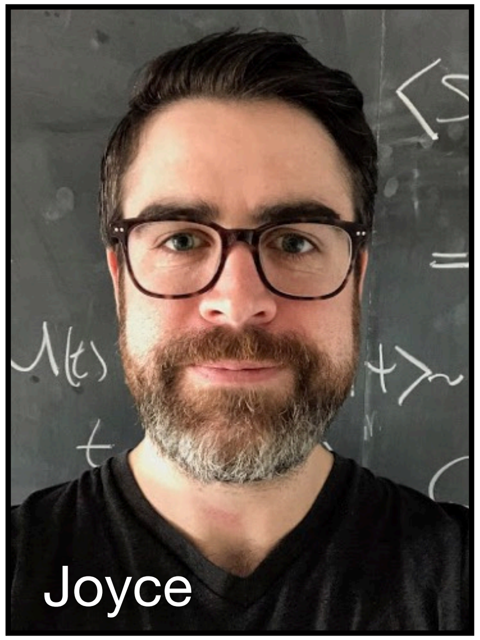
**Daniel Baumann**

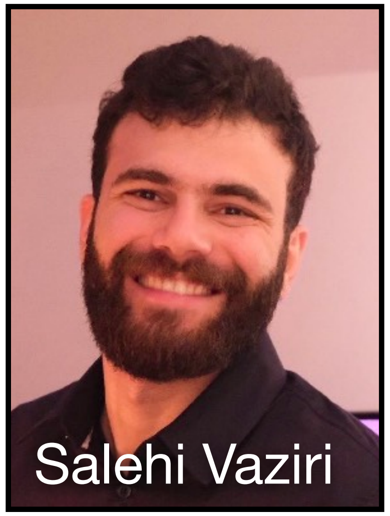
University of Amsterdam &
National Taiwan University

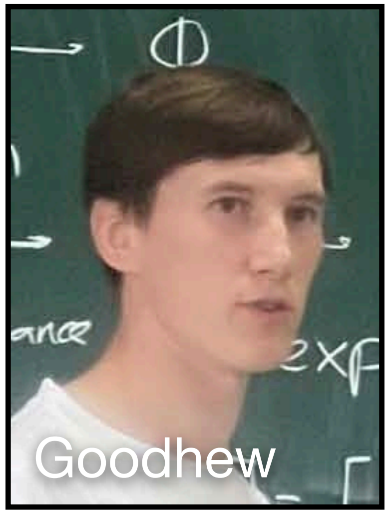
**UNIVERSE+ Seminar**

# Based on work with







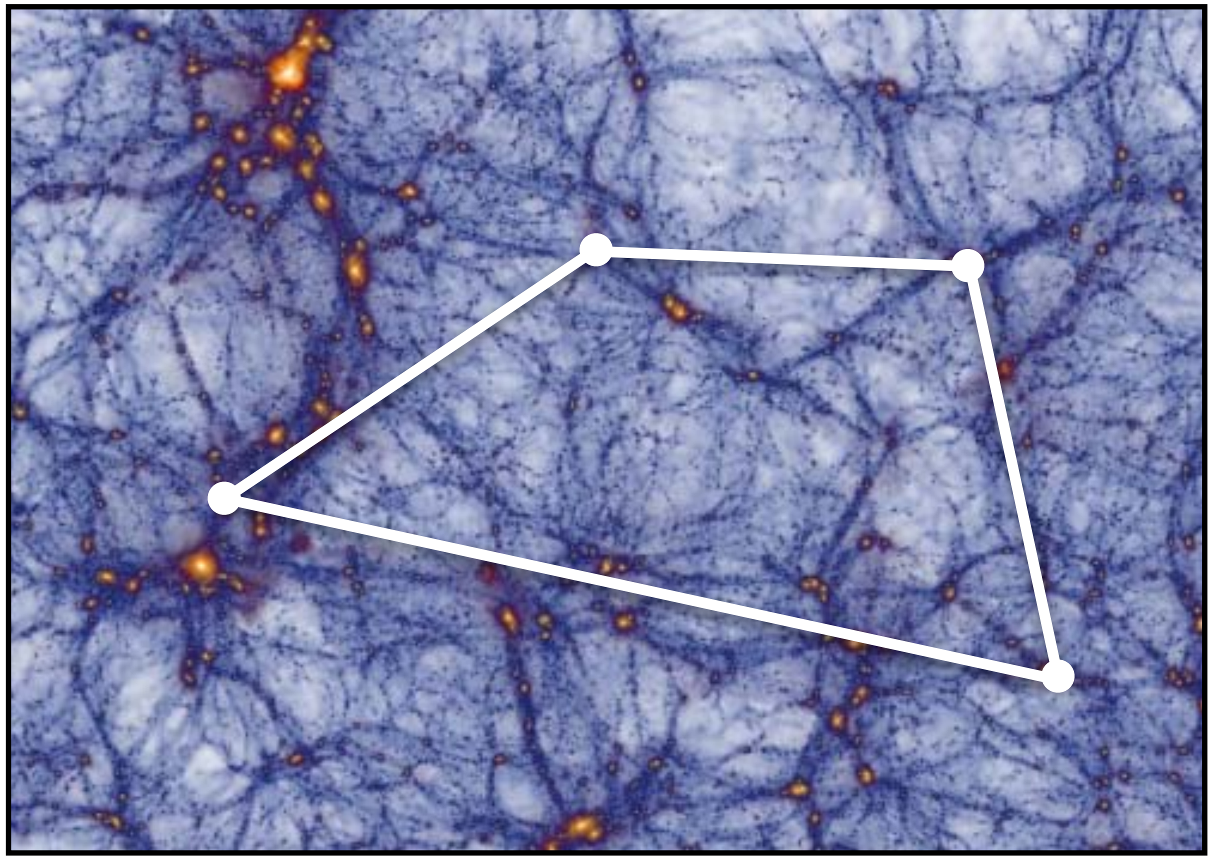


*Kinematic Flow and the Emergence of Time* [arXiv:2312.05300]

*Differential Equations for Cosmological Correlators* [arXiv:2312.05303]

# Motivation

Correlation functions are the main observables in cosmology:



$$= \langle \delta\rho(\vec{x}_1)\delta\rho(\vec{x}_2)\cdots\delta\rho(\vec{x}_N)\rangle$$

They encode the history of the universe. (See my Colloquium in Leipzig)

# Motivation

In this talk, we will consider a toy model of cosmology:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \frac{\lambda}{3!}\phi^3\right] \qquad a(t) \propto \frac{1}{t^{1+\varepsilon}}$$

Conformal mass

Non-conformal interaction

$\varepsilon = 0 :$ dS
$\varepsilon = -1 :$ flat
$\varepsilon = -2 :$ radiation
$\varepsilon = -3 :$ matter

This allows us to derive a large amount of “mathematical data” and look for hidden patterns in the results.

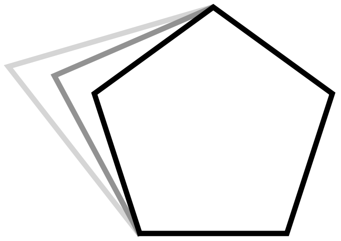
# Motivation

Correlators in this theory can be written as twisted integrals:

$$\psi(X_i, Y_j) = \int_0^\infty \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n \, \underbrace{(\omega_1 \cdots \omega_n)^\varepsilon}_{\text{Twist}} \, \underbrace{\psi_{\text{flat}}(X_i + \omega_i, Y_j)}_{\text{Flat-space wavefunction}}$$

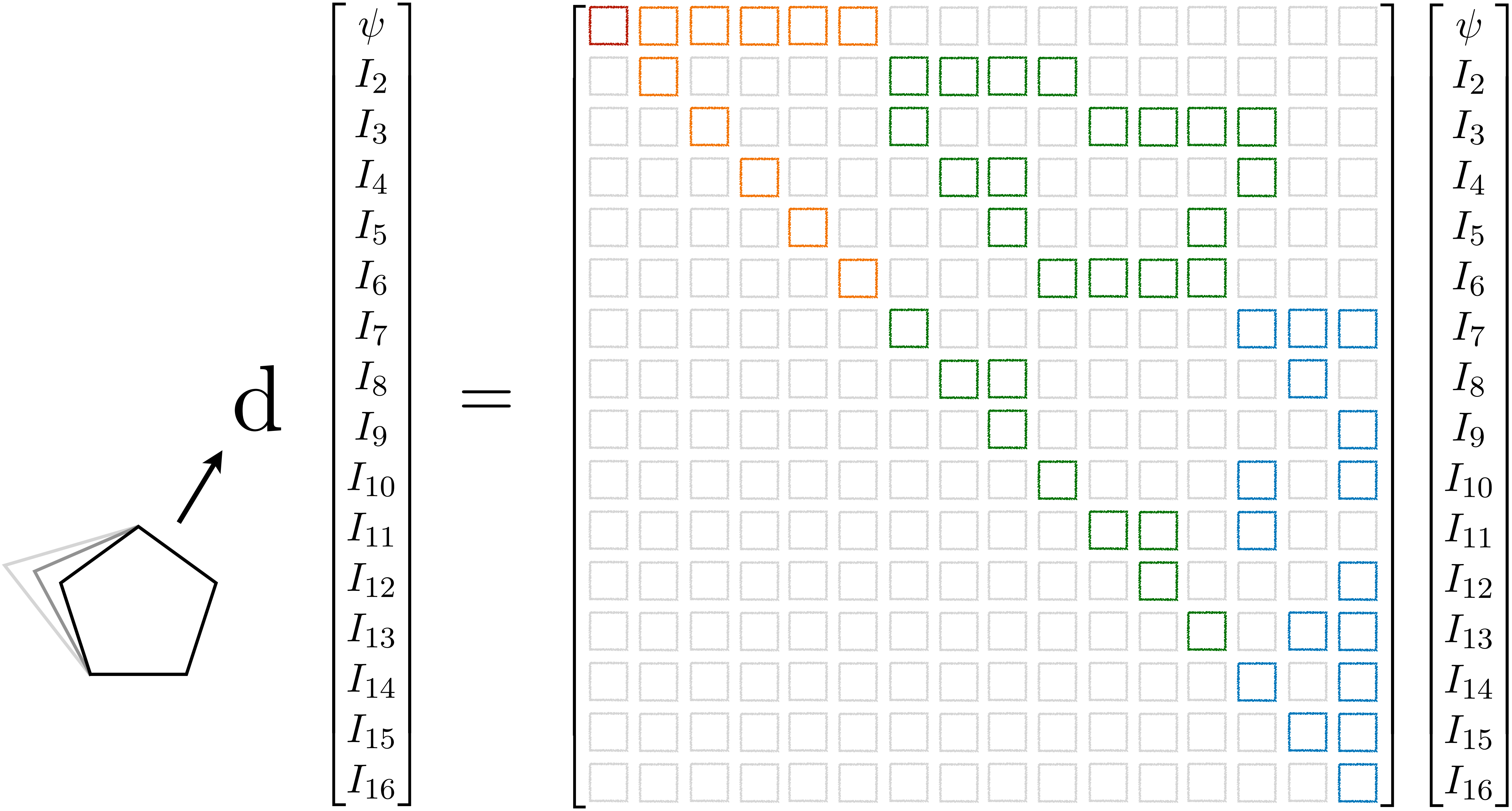
They are part of a vector space of master integrals:

$$\vec{I} \equiv \begin{bmatrix} \psi \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix} \xrightarrow{\qquad\qquad} \mathrm{d}\vec{I} = A\vec{I}$$



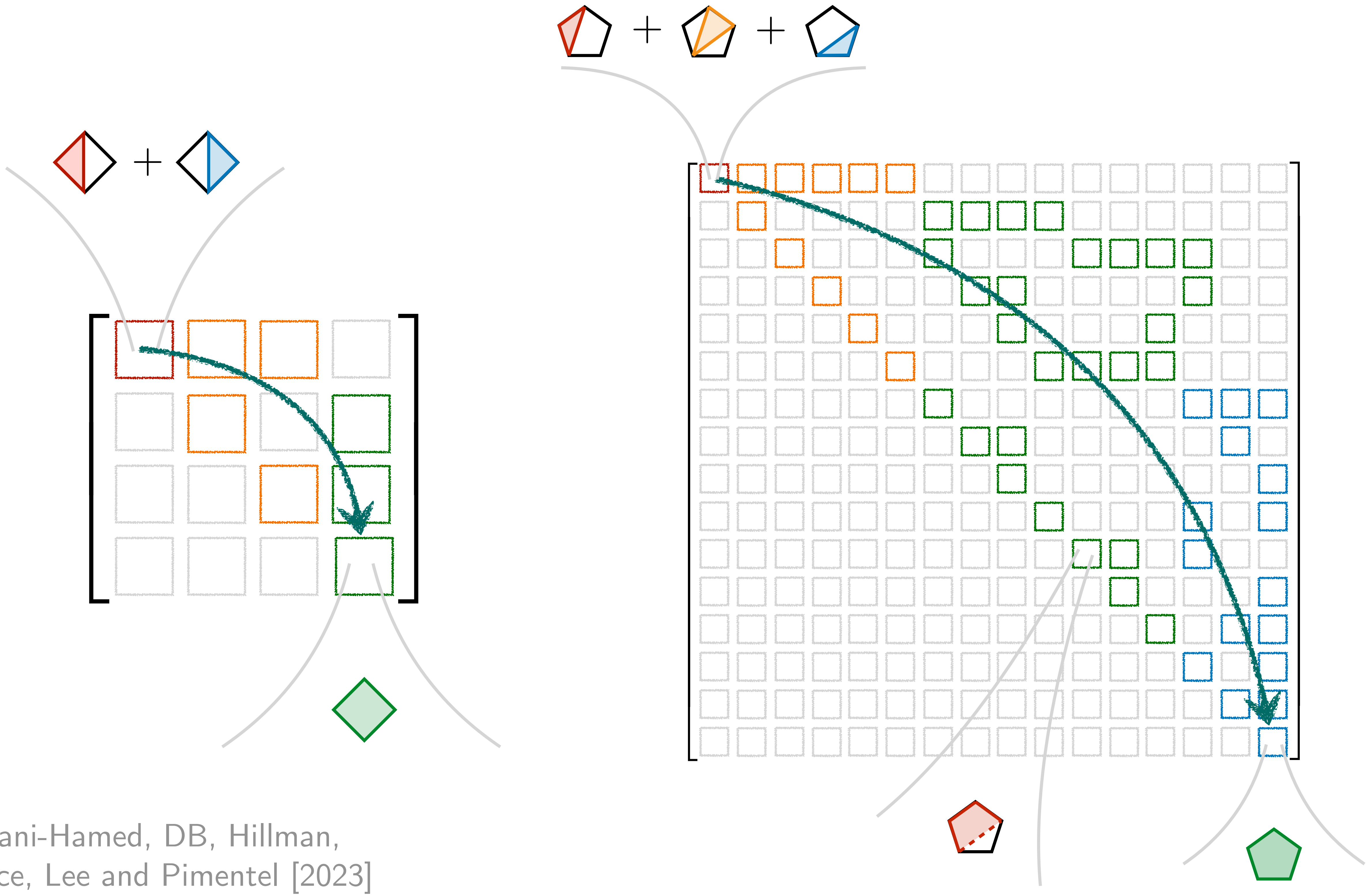
# Motivation

The differential equations satisfied by the master integrals quickly become very complex:



# Motivation

Something remarkable happened when we drew pictures of the results!



Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [2023]

# Outline

Background

Differential Equations

A Hidden Pattern

Conclusions

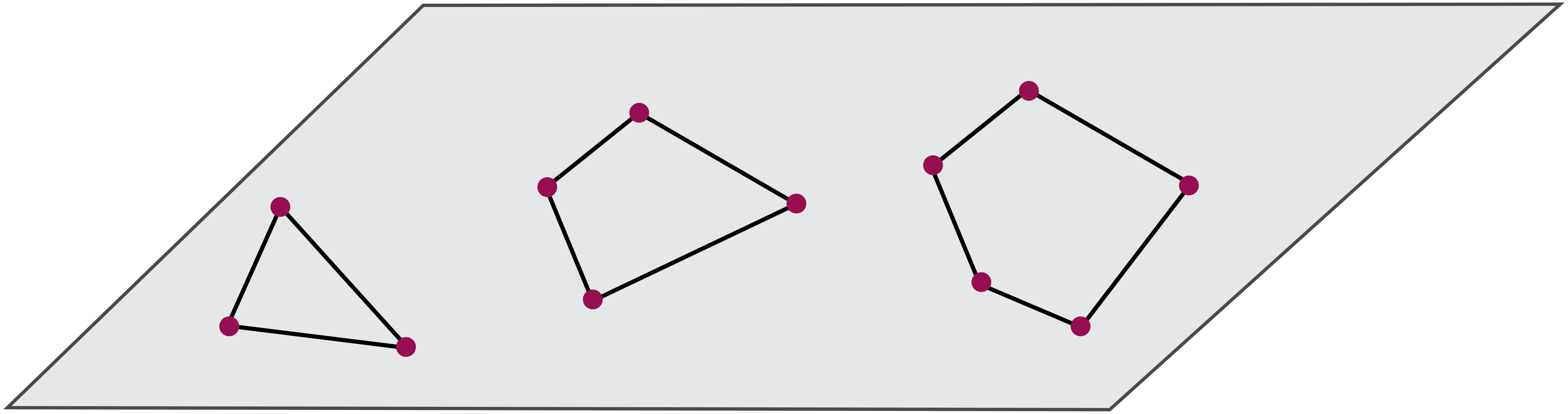
# Background

# Wavefunction

Correlators can be computed in terms of a wavefunction:

$$\langle \varphi(\vec{x}_1)\dots\varphi(\vec{x}_N)\rangle = \int \mathcal{D}\varphi\ \varphi(\vec{x}_1)\dots\varphi(\vec{x}_N)\ |\Psi[\varphi]|^2$$

Probability

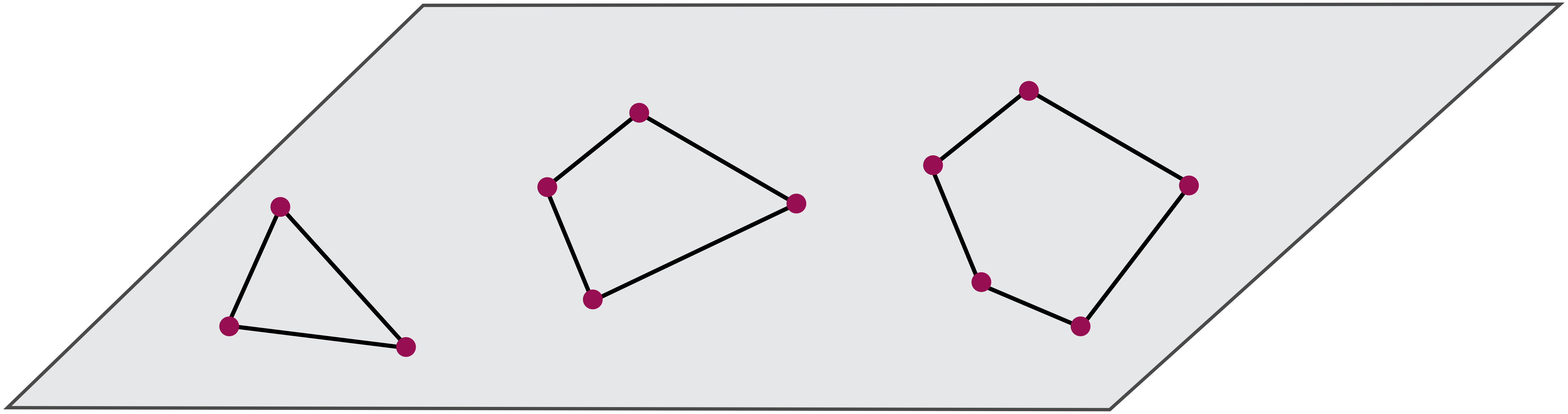


# Wavefunction

For small fluctuations, we expand the wavefunction as

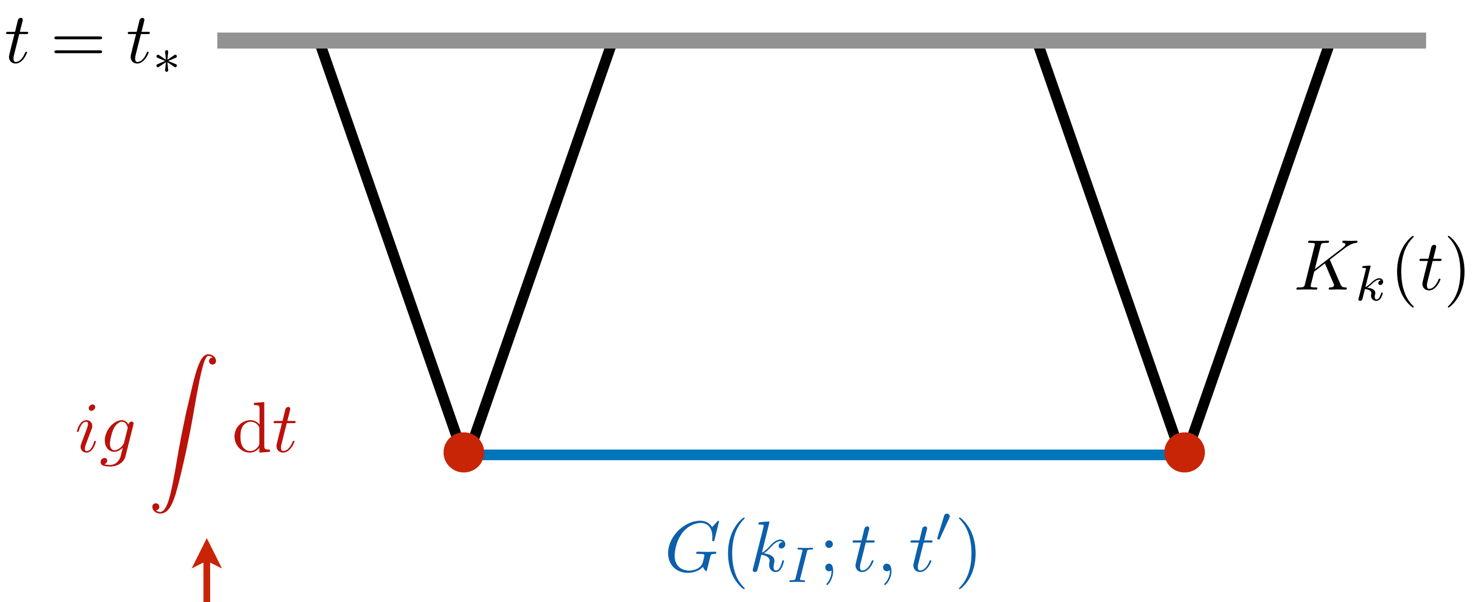
$$\Psi[\varphi] = \exp\left( -\sum_N \int \frac{\mathrm{d}^3k_1 \ldots \mathrm{d}^3k_N}{(2\pi)^{3N}} \, \psi_N(\vec{k}_1, \cdots, \vec{k}_N) \, \varphi_{\vec{k}_1} \ldots \varphi_{\vec{k}_N} \right)$$

Wavefunction coefficient



# Feynman Rules

The wavefunction coefficients are determined by simple Feynman rules:



$t = t_*$

$K_k(t)$

**Bulk-to-boundary propagator** for every external line

$ig \int \mathrm{d}t$

**Time integral** for every vertex

$G(k_I; t, t')$

**Bulk-to-bulk propagator** for every internal line

---

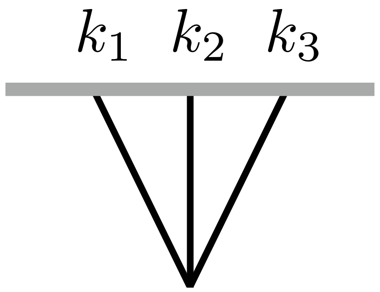
$$K_k(t) = \frac{\phi_k(t)}{\phi_k(t_*)}$$

**Mode function**

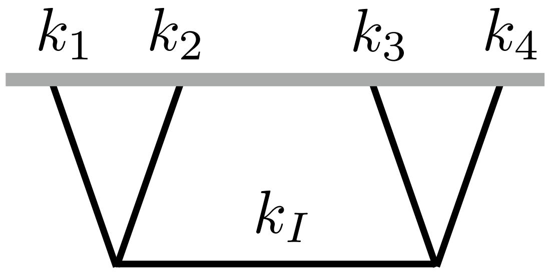
$$G_k(t,t') = \phi_k^*(t)\phi_k(t')\,\theta(t-t') + \phi_k^*(t')\phi_k(t)\,\theta(t'-t) - \frac{\phi_k^*(t_*)}{\phi_k(t_*)}\phi_k(t)\phi_k(t')$$

# Flat Space

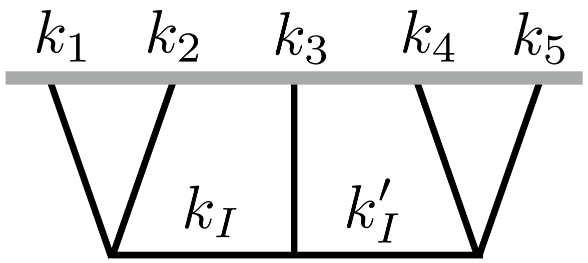
In flat space, it is easy to compute these correlators: $\phi_k(t) = e^{ikt}$



$$= ig\int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t} = \frac{g}{k_1+k_2+k_3}$$



$$= \frac{g^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}$$

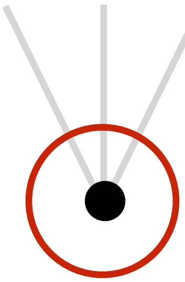


$$= \frac{g^3}{k_{12345}(k_{12}+k_I)(k_3+k_I+k_I')(k_{45}+k_I')}\left[\frac{1}{k_{123}+k_I'}+\frac{1}{k_{345}+k_I}\right]$$

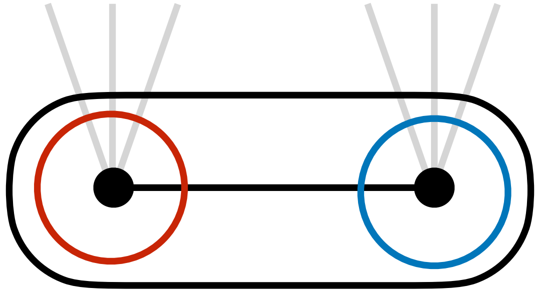
Results are rational functions of the energies entering each vertex.

# Graph Tubings

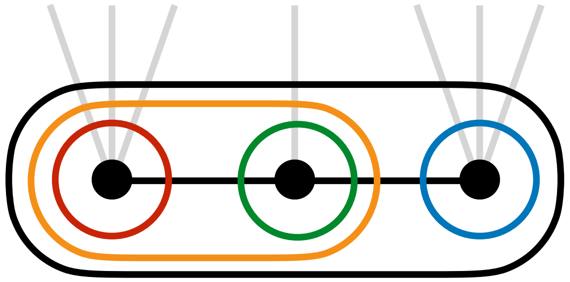
We can represent the results by graph tubings: $\psi^{\text{flat}} = \sum_{\mathcal{T}} \prod_a \frac{1}{E_a}$



$$= \frac{1}{X}$$



$$= \frac{1}{(X_1 + X_2)(X_1 + Y)(X_2 + Y)}$$



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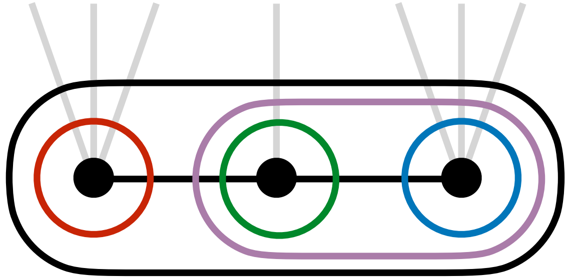
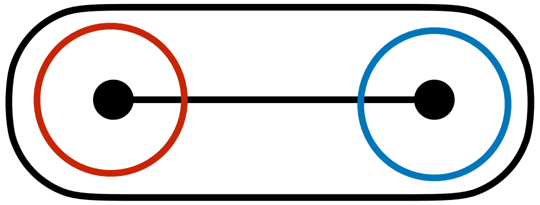


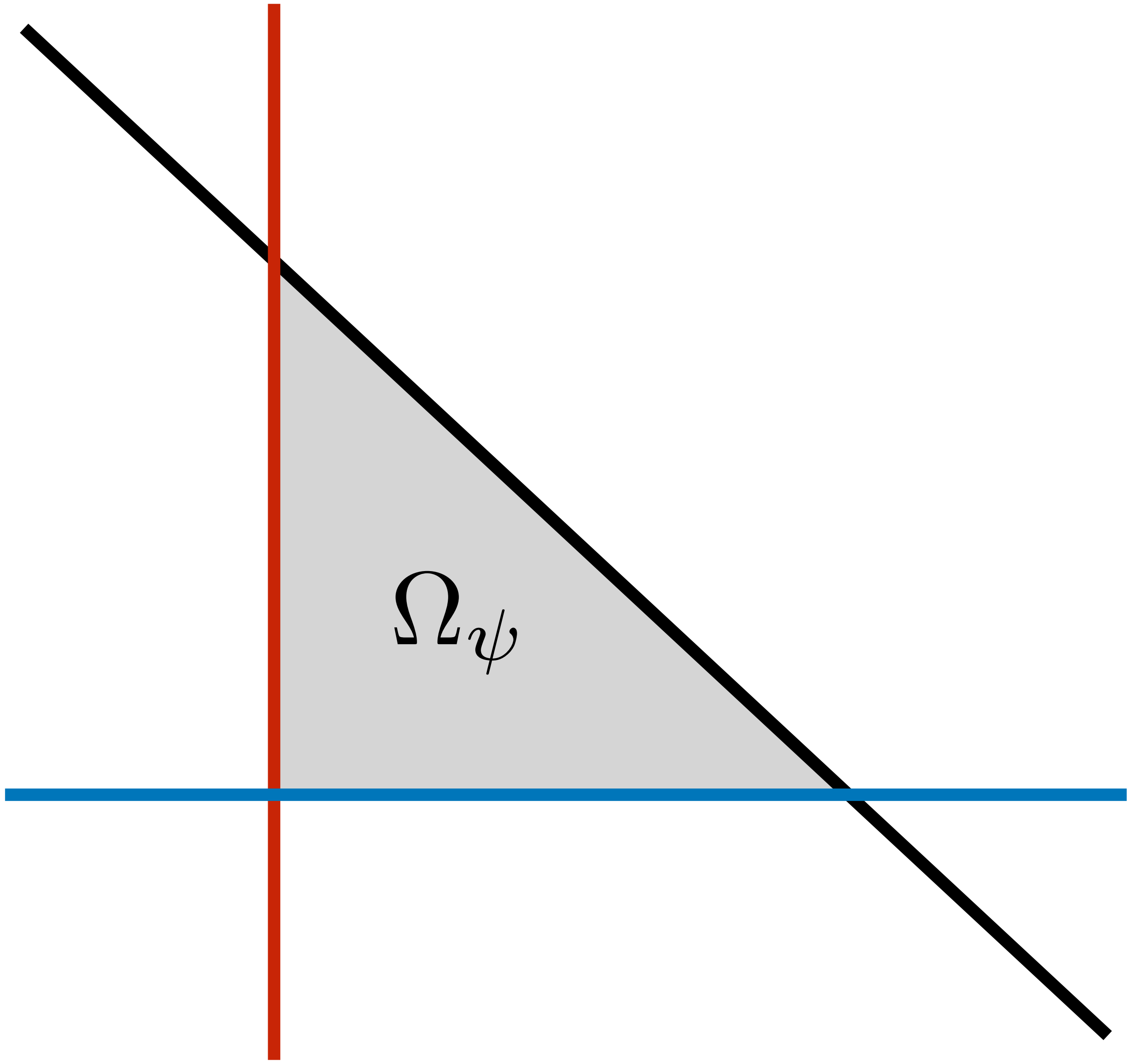
$$= \frac{1}{(X_{123})(X_{12} + Y)(X_3 + Y + Y')(X_{45} + Y')} \left[ \frac{1}{X_{12} + Y'} + \frac{1}{X_{23} + Y} \right]$$

Arkani-Hamed, Benincasa and Postnikov [2017]

# Canonical Forms

The results also correspond to canonical forms of the regions bounded by the singular lines:





"cosmological polytopes"

Arkani-Hamed, Benincasa and Postnikov [2017]

# Power-Law Cosmology

We will consider a toy model of cosmology:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \frac{\lambda}{3!}\phi^3\right] \qquad a(t) \propto \frac{1}{t^{1+\varepsilon}}$$

Conformal mass

Non-conformal interaction

$\varepsilon = 0:$ dS

$\varepsilon = -1:$ flat

$\varepsilon = -2:$ radiation

$\varepsilon = -3:$ matter

Mode functions are still simple: $\phi_k(t) = t^{1+\varepsilon}e^{ikt}$

We can relate the correlators in this theory to the flat-space results.

# Correlators as Twisted Integrals

Using

$$t^{-(1+\varepsilon)} = -\frac{ie^{-\frac{i\pi\varepsilon}{2}}}{\Gamma(1+\varepsilon)} \int_0^\infty \mathrm{d}\omega\, \omega^\varepsilon e^{i\omega t}$$

we find

$$\psi(X_i, Y_j) = \int_0^\infty \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n \, (\omega_1 \cdots \omega_n)^\varepsilon \, \psi_{\text{flat}}(X_i + \omega_i, Y_j)$$

Twist

Flat-space wavefunction

We will study these twisted integrals using the method of differential equations.

Kotikov [1991]
Remiddi [1997]
Henn [2012]

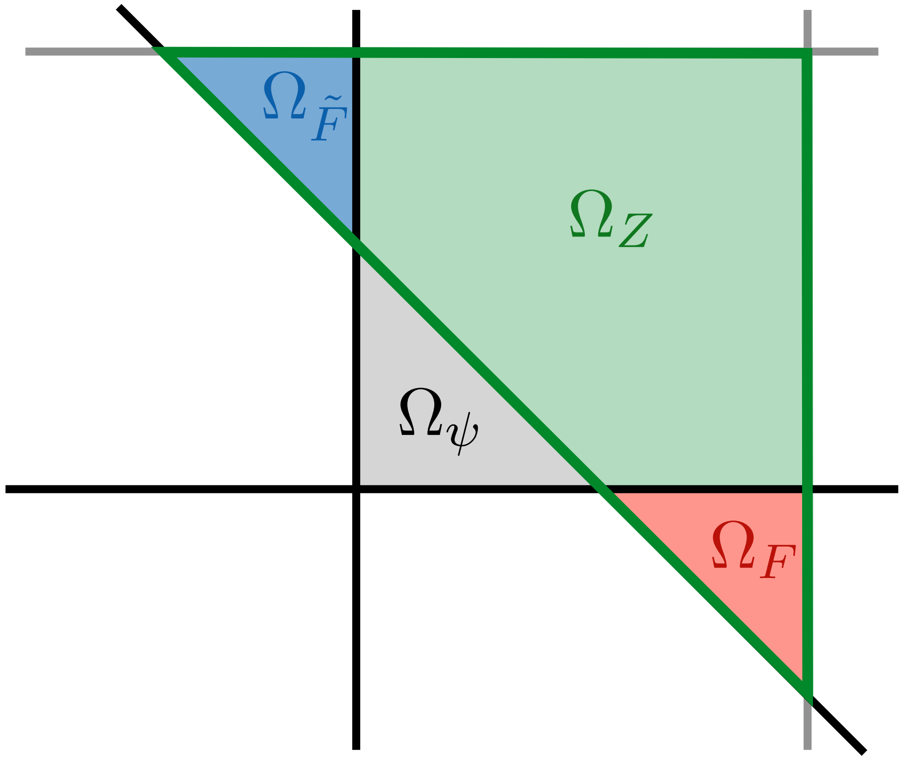
# Differential Equations

# Master Integrals

Introduce a family of integrals with the same singularities:

$$I_N = \int_0^\infty \mathrm{d}\omega_1 \mathrm{d}\omega_2 \, \frac{\omega_1^{\varepsilon - n_1} \omega_2^{\varepsilon - n_2}}{(X_1 + X_2 + \omega_1 + \omega_2)^{n_3} (X_1 + \omega_1 + Y)^{n_4} (X_2 + \omega_2 + Y)^{n_5}}$$

- These integrals form a finite-dimensional vector space.
- Number of master integrals = number of bounded regions:



$$\vec{I} = \begin{bmatrix} \psi \\ F \\ \tilde{F} \\ Z \end{bmatrix} = \int (\omega_1 \omega_2)^\varepsilon \begin{bmatrix} \Omega_\psi \\ \Omega_F \\ \Omega_{\tilde{F}} \\ \Omega_Z \end{bmatrix}$$

- A preferred basis is given by the canonical forms.

Aomoto, Kita [2011]
Mastrolia, Mizera [2018]

# Differential Equations

Being part of a finite-dimensional vector space, the master integrals satisfy coupled differential equations:

$$\boxed{\mathrm{d}\vec{I} = \varepsilon A\,\vec{I}}$$

$$\mathrm{d} = \sum_i \mathrm{d}X_i \frac{\partial}{\partial X_i} \qquad\qquad A = \sum_n \alpha_n\, \mathrm{d}\log \Phi_n(X_i)$$

**Letters**

For the two-site chain, we have

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathrm{d}\log(X_1 + X_2) + \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 + Y) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 - Y)$$

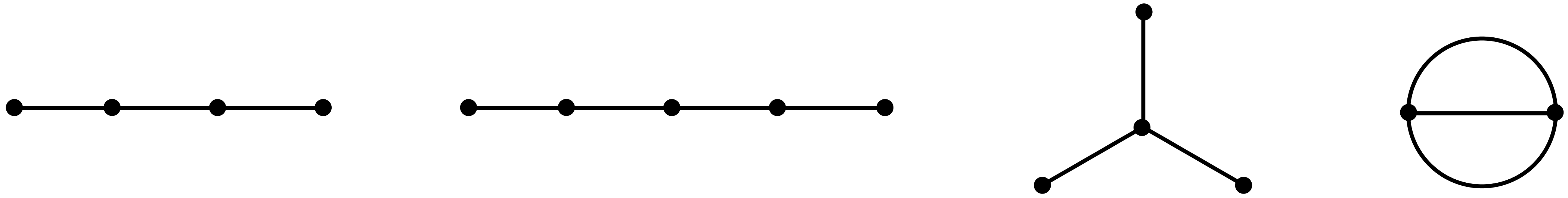
$$+ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 + Y) + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 - Y)$$

De and Pokraka [2023]

Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [2023]

# Increasing Complexity

This approach breaks down for more complicated graphs:



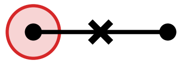
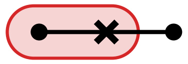
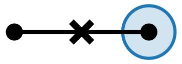
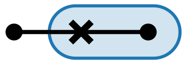
- Hard to visualize the higher-dimensional integrals.
- Finding an optimal basis is a bit of an art.
- Finding the differential equations is algebraically challenging.
- Results aren't very enlightening.

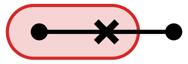
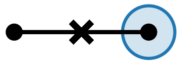
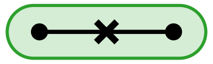
Remarkably, there are hidden **combinatorial** and **geometric structures** underlying these differential equations that allow us to bypass these challenges.

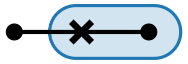
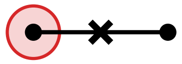
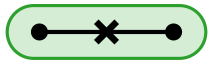
# A Hidden Pattern

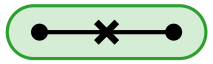
# Graphical Representation

The differential equations for the two-site chain can be written as

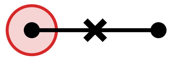
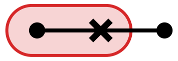
$\mathrm{d}\psi = \varepsilon\,[(\psi - F)$  $+\ F$  $+\ (\psi - \tilde{F})$  $+\ \tilde{F}$  $]$

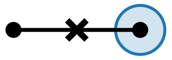
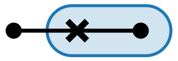
$\mathrm{d}F = \varepsilon\,[F$  $+\ (F - Z)$  $+\ Z$  $]$

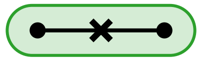
$\mathrm{d}\tilde{F} = \varepsilon\,[\tilde{F}$  $+\ (\tilde{F} - Z)$  $+\ Z$  $]$

$\mathrm{d}Z = 2\varepsilon\, Z$ 

where

 $\equiv \mathrm{d}\log(X_1 + Y)$  $\equiv \mathrm{d}\log(X_1 - Y)$

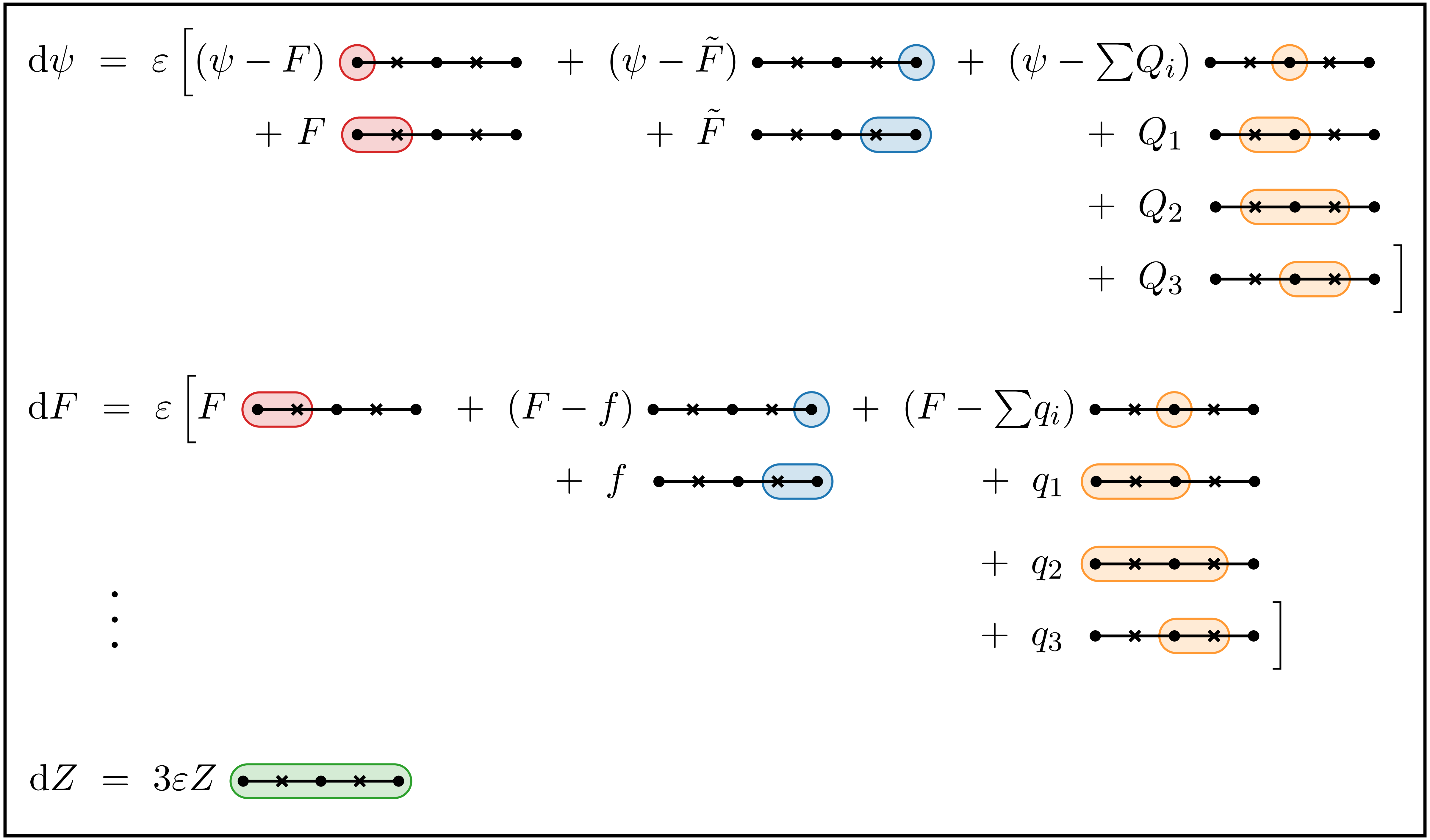
 $\equiv \mathrm{d}\log(X_2 + Y)$  $\equiv \mathrm{d}\log(X_2 - Y)$

 $\equiv \mathrm{d}\log(X_1 + X_2)$

**Tubings of marked graphs**

# Kinematic Flow

Upon taking derivatives, the graph tubings grow:



$$\mathrm{d}\psi = \varepsilon\Big[(\psi - F) \quad + \quad (\psi - \tilde{F}) \quad + \quad (\psi - \textstyle\sum Q_i)$$

$$+\ F \qquad + \ \tilde{F} \qquad + \ Q_1$$

$$+ \ Q_2$$

$$+ \ Q_3 \ \Big]$$

$$\mathrm{d}F = \varepsilon\Big[F \quad + \quad (F - f) \quad + \quad (F - \textstyle\sum q_i)$$

$$+ \ f \qquad + \ q_1$$

$$+ \ q_2$$

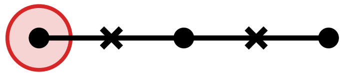
$$\vdots \qquad + \ q_3 \ \Big]$$

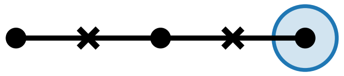
$$\mathrm{d}Z = 3\varepsilon Z$$

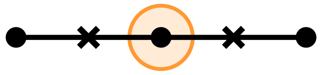
Simple rules allow us to predict this “evolution” for arbitrary graphs.

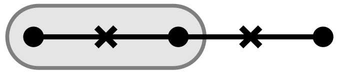
# Letters

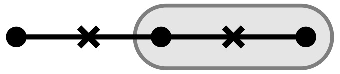
Letters are **connected tubes** of marked graphs:

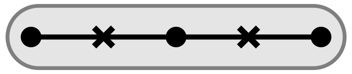

$\equiv \mathrm{d}\log(X_1 + Y)$


$\equiv \mathrm{d}\log(X_3 + Y')$

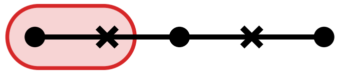

$\equiv \mathrm{d}\log(X_2 + Y + Y')$

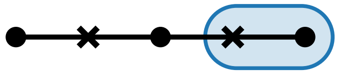

$\equiv \mathrm{d}\log(X_1 + X_2 + Y')$

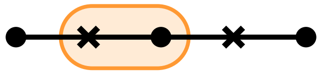

$\equiv \mathrm{d}\log(X_2 + X_3 + Y)$

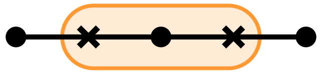

$\equiv \mathrm{d}\log(X_1 + X_2 + X_3)$

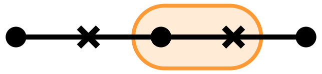
Singularities of the integrand

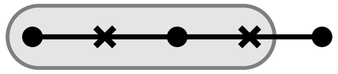

$\equiv \mathrm{d}\log(X_1 - Y)$

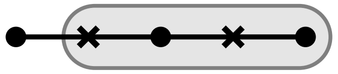

$\equiv \mathrm{d}\log(X_3 - Y')$


$\equiv \mathrm{d}\log(X_2 - Y + Y')$


$\equiv \mathrm{d}\log(X_2 - Y - Y')$


$\equiv \mathrm{d}\log(X_2 + Y - Y')$

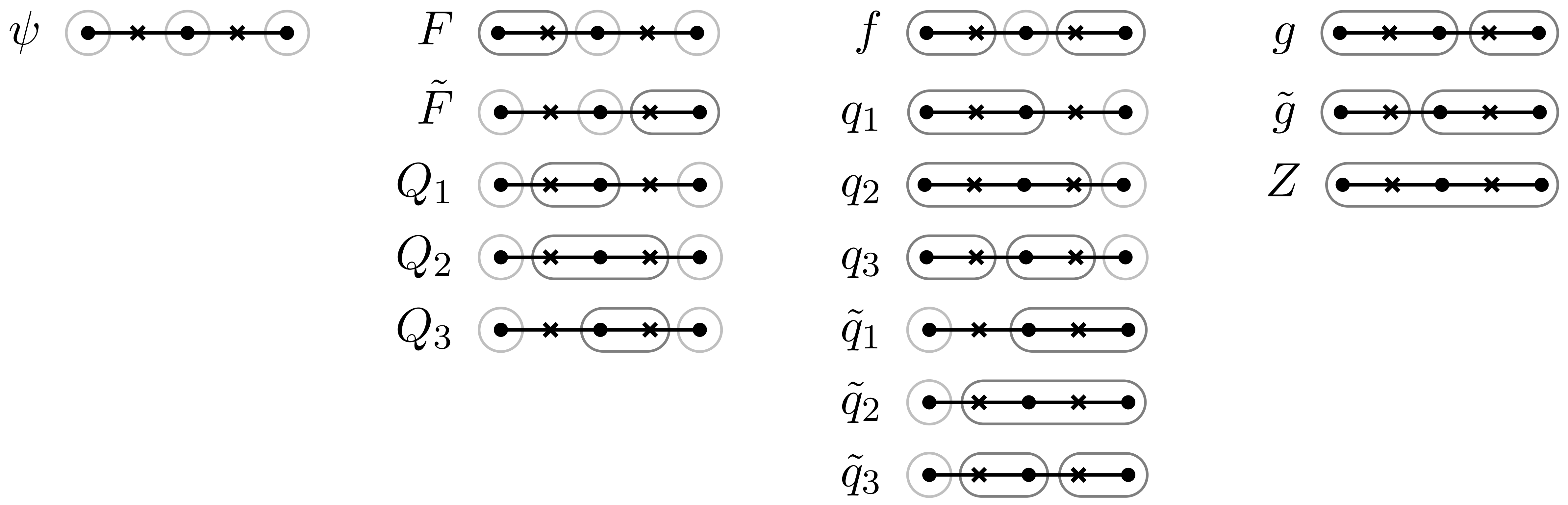

$\equiv \mathrm{d}\log(X_1 + X_2 - Y')$


$\equiv \mathrm{d}\log(X_2 + X_3 - Y')$

Extra singularities

# Functions

Functions are **complete tubings** of marked graphs:



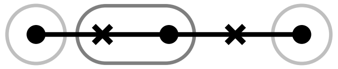
- Uniquely defines a basis of functions.
- Each function is related to the original wavefunction by replacement rules:

$$\frac{1}{E_1E_2E_3}\frac{1}{E}[\cdots]$$

$$\frac{1}{\omega_1 E_2E_3}\frac{1}{E}[\cdots]$$

# Kinematic Flow

**1.** Start with the graph tubing associated to a **parent function** of interest:

$Q_1$ 

Arkani-Hamed, DB, Hillman,
Joyce, Lee and Pimentel [2023]

# Kinematic Flow

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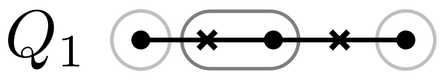
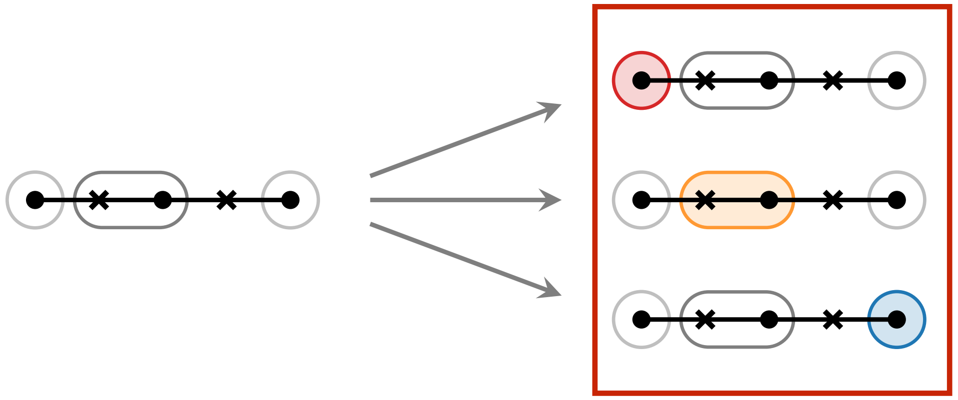


**2.** Generate a family tree of its **descendants**:



**Activation**

- **Activate** the tube enclosing each vertex.

Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [2023]

# Kinematic Flow

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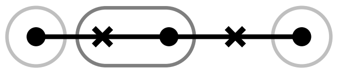

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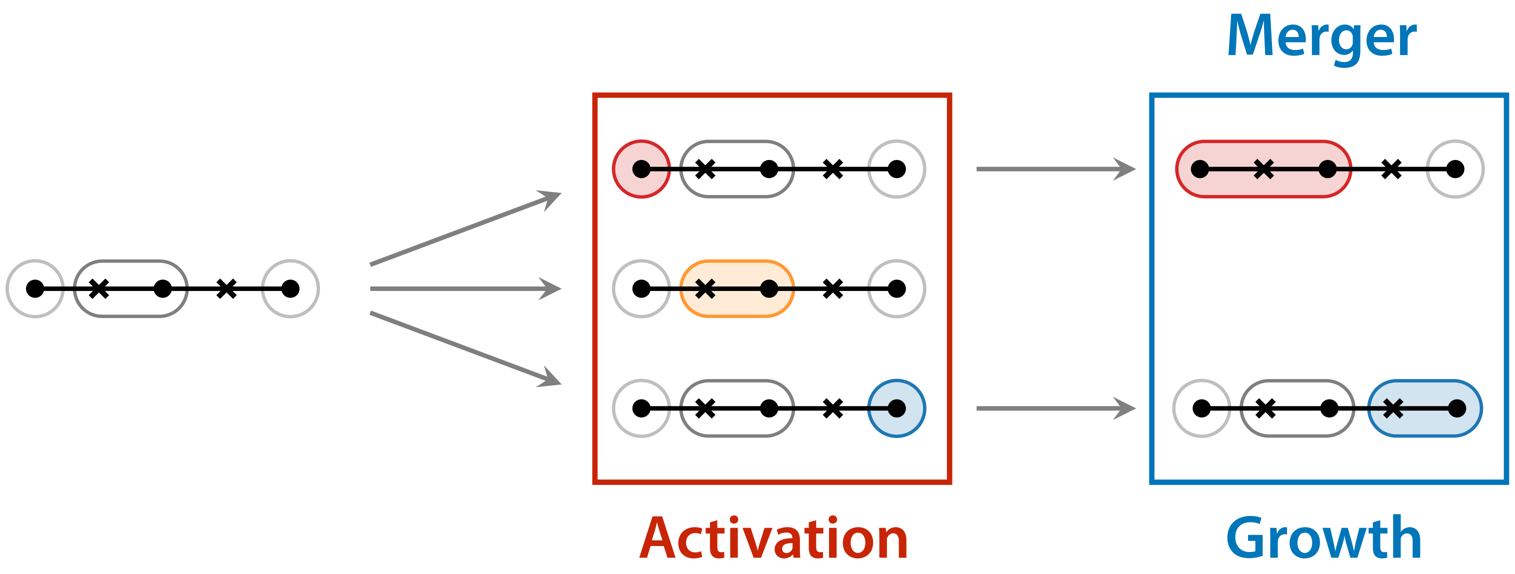

- **Activate** the tube enclosing each vertex.
- Activated tubes can **grow** to enclose adjacent crosses.
- If the grown tube intersects another tube, they **merge**.

Arkani-Hamed, DB, Hillman,
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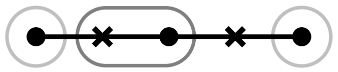
$Q_1$ 

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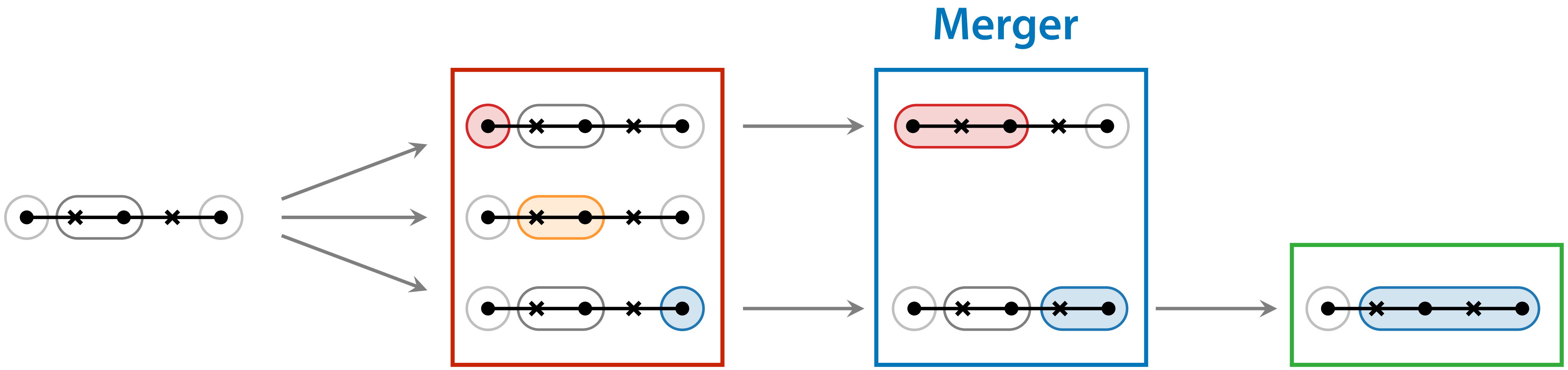


Merger

Activation

Growth

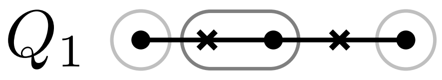
Absorption

- **Activate** the tube enclosing each vertex.
- Activated tubes can **grow** to enclose adjacent crosses.
- If the grown tube intersects another tube, they **merge**.
- If an activated tube is adjacent to another tube, it can **absorb** it.

Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [2023]

# Kinematic Flow

**1.** Start with the graph tubing associated to a **parent function** of interest:



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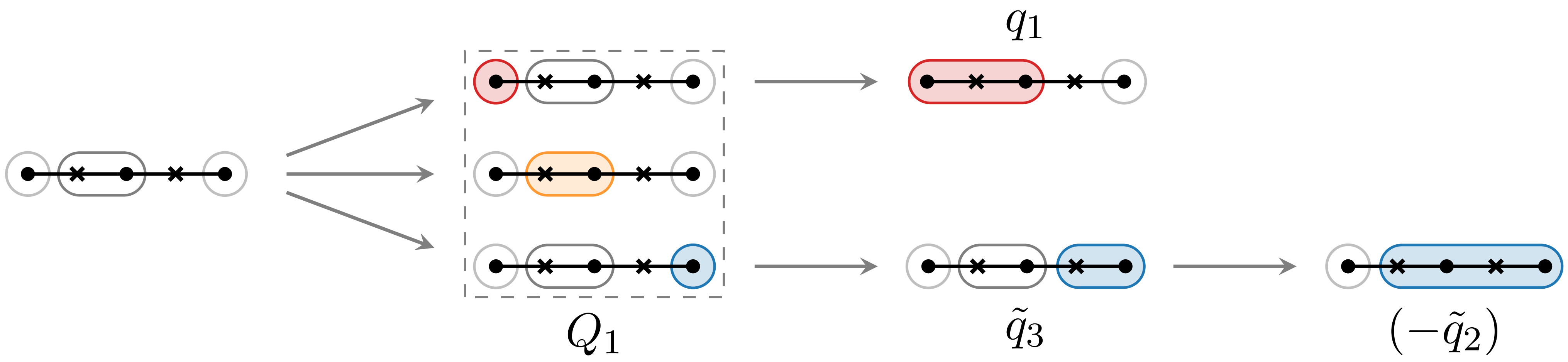
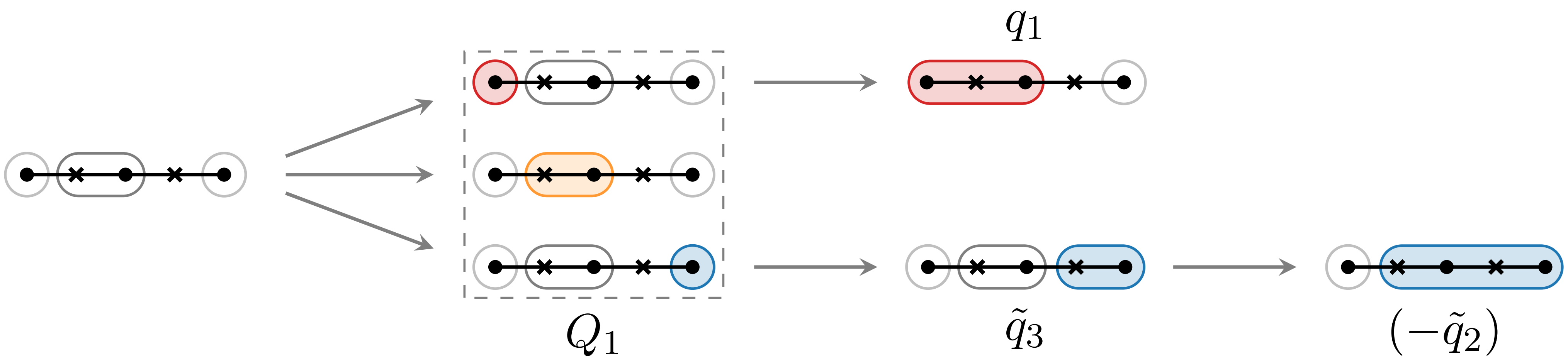


**3.** Assign functions to each graph tubing in the tree.

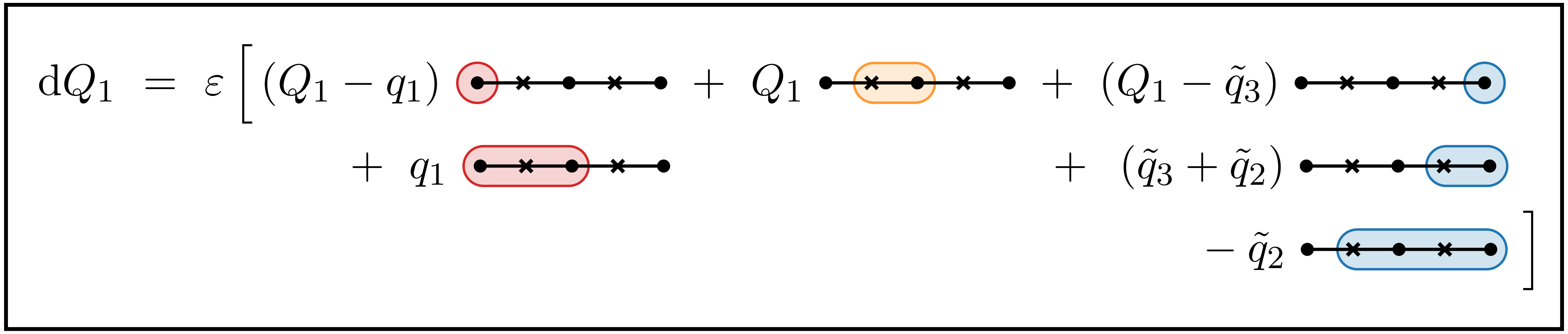
Arkani-Hamed, DB, Hillman,
Joyce, Lee and Pimentel [2023]

# Kinematic Flow

**4.** From the family tree, we directly read off the differential equation:



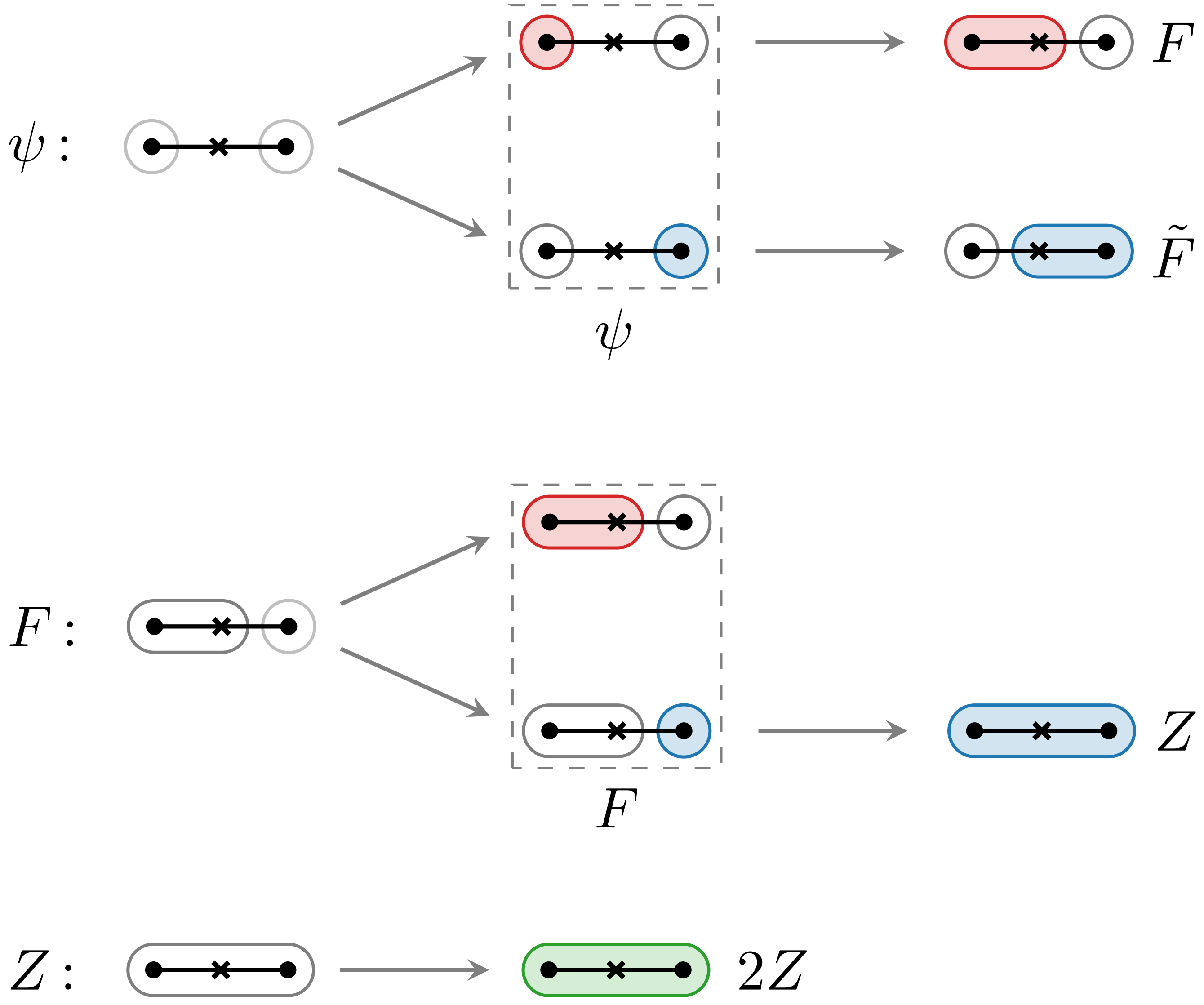
- Each activated tube becomes a **letter** in the differential equation.
- The **coefficient** of each letter is the function associated to the graph *minus* the functions associated to its immediate descendant graphs.



Remarkably, this works for arbitrary tree graphs and loop integrands!

# Examples

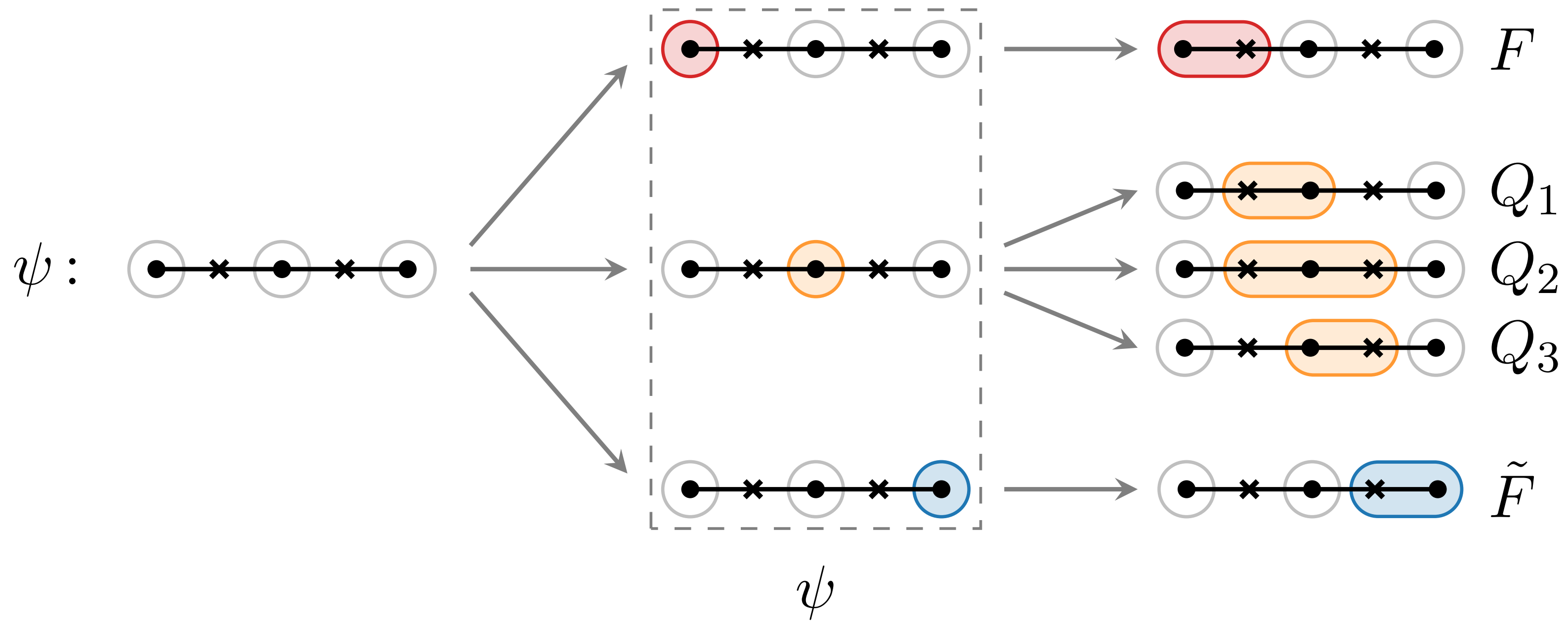
The equations for the **two-site chain** follow from:

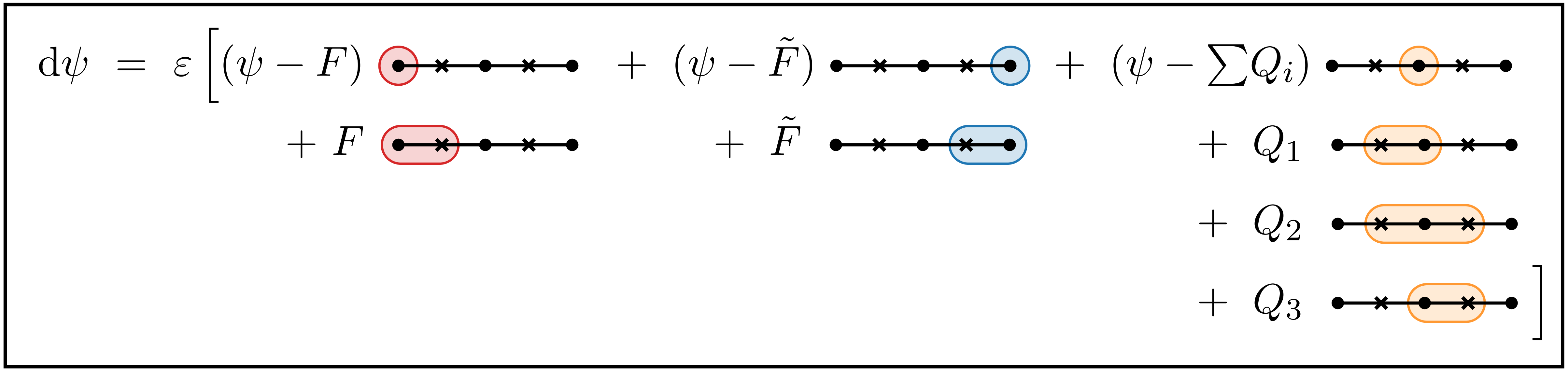


Arkani-Hamed, DB, Hillman,
Joyce, Lee and Pimentel [2023]

# Examples

The equations for the **three-site chain** follow from:





Arkani-Hamed, DB, Hillman,
Joyce, Lee and Pimentel [2023]

# Examples

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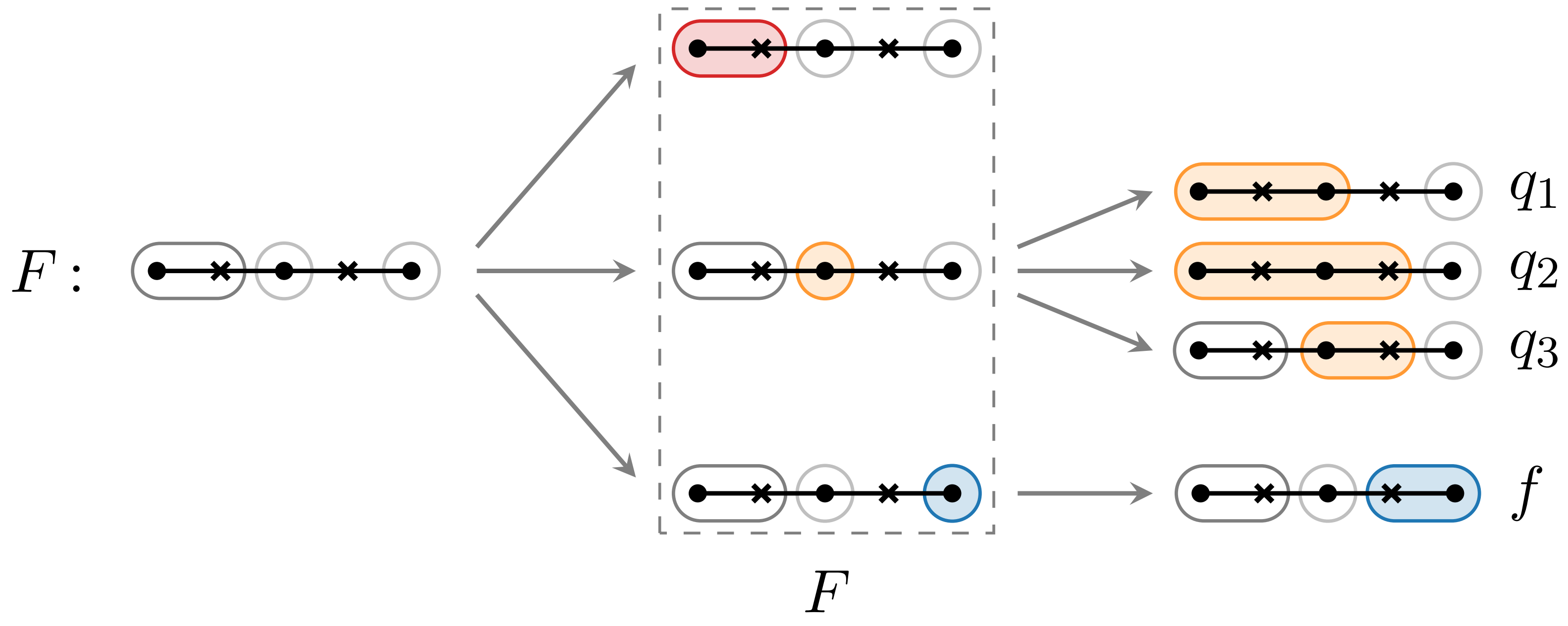

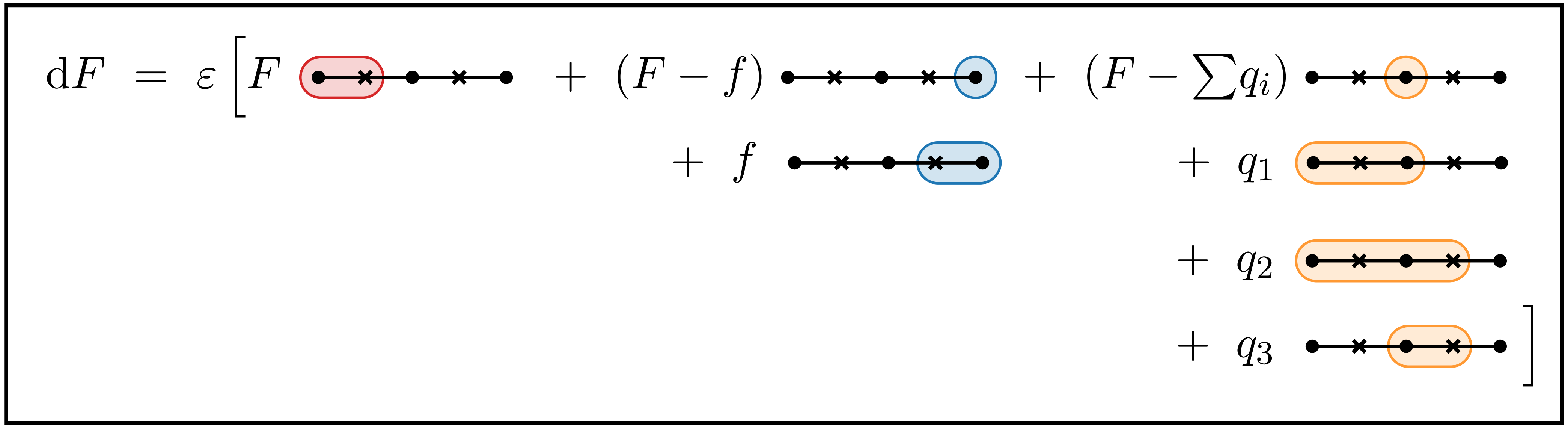

Arkani-Hamed, DB, Hillman,
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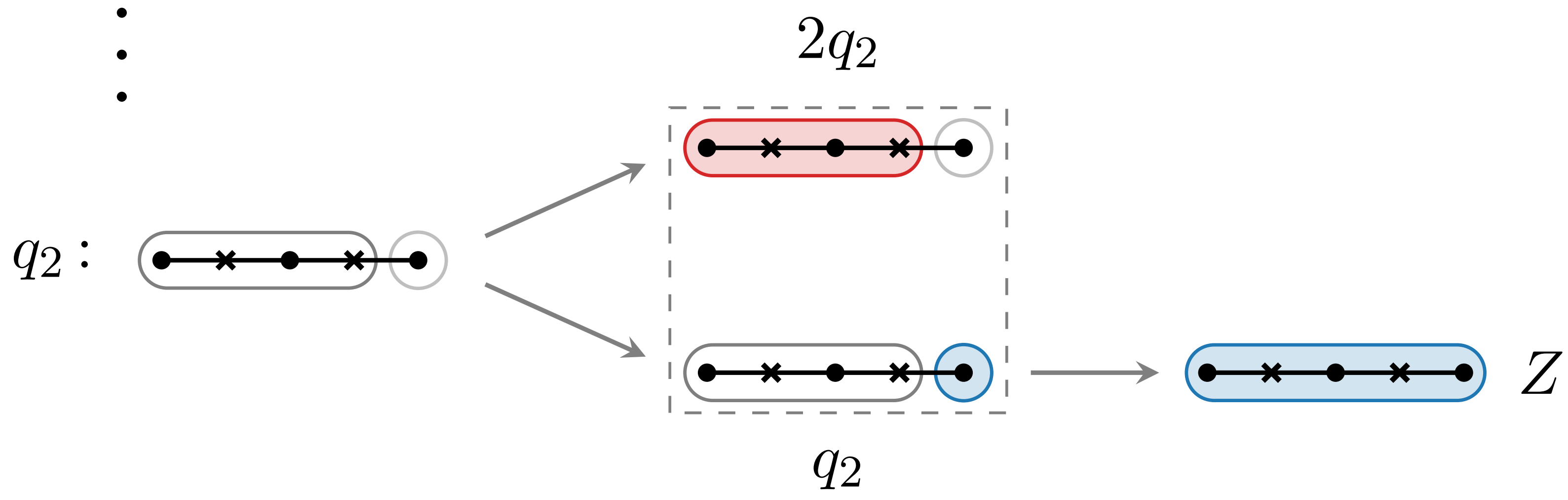
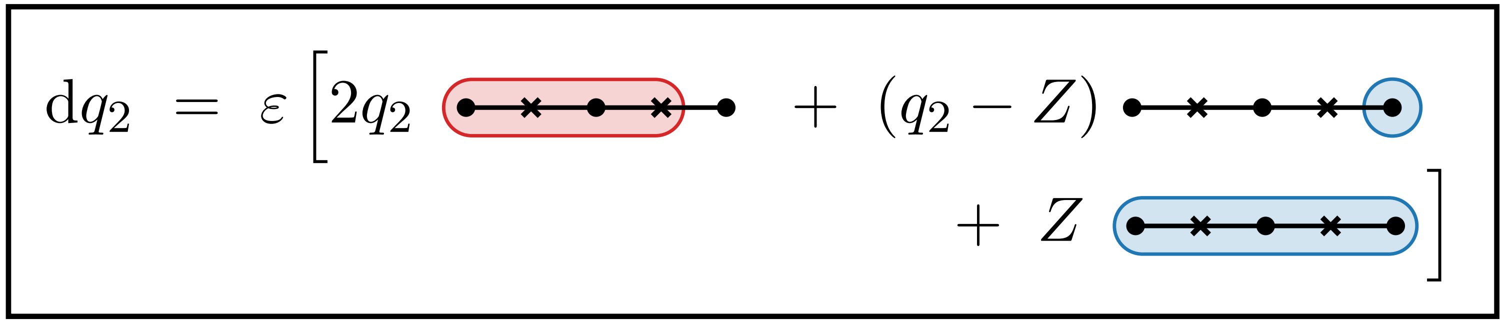
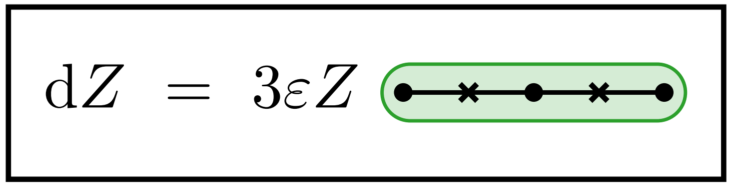


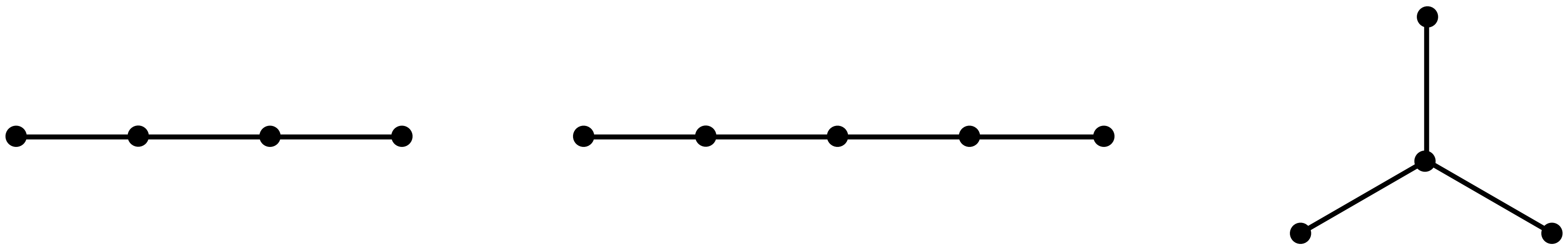




Arkani-Hamed, DB, Hillman,
Joyce, Lee and Pimentel [2023]

# Examples

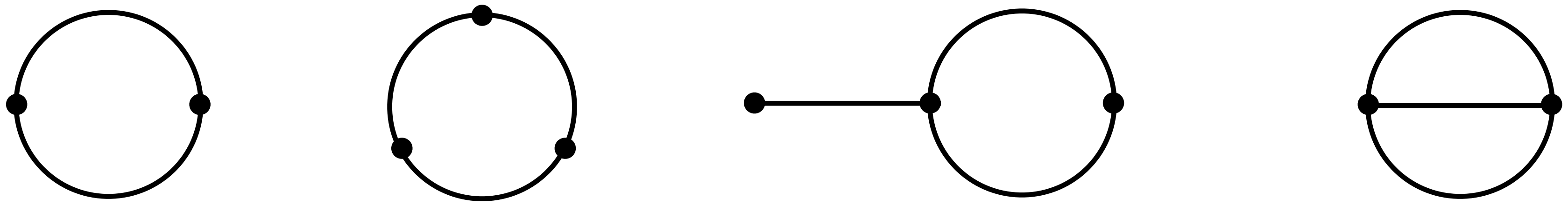
The graphical rules are local and can be used to predict the differential equations for **arbitrary tree graphs** with different topologies:



In the paper, we present many nontrivial examples.

Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [2023]

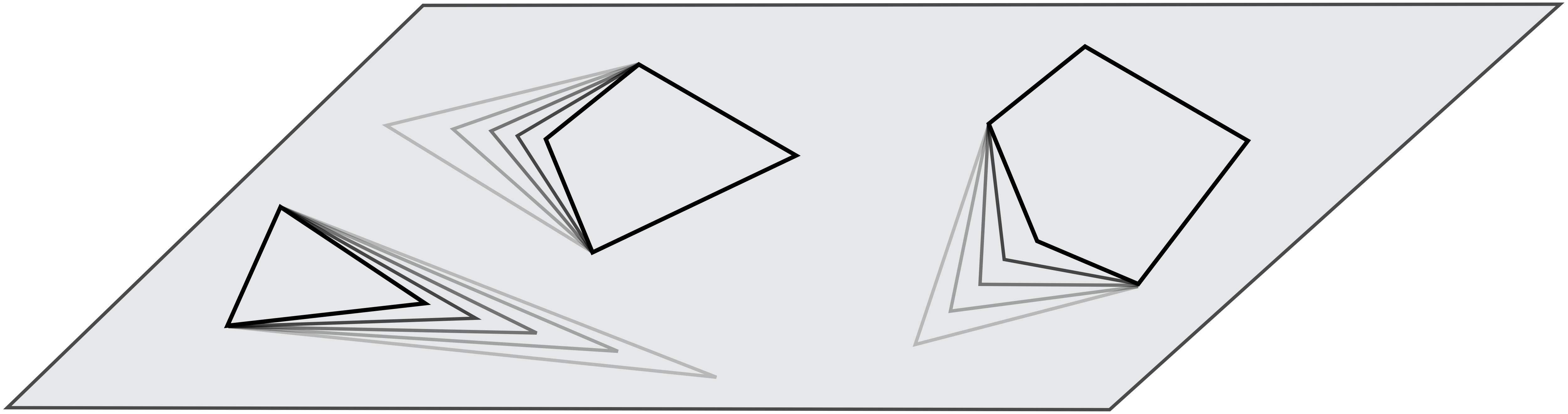
The same rules also work for **loop integrands**:

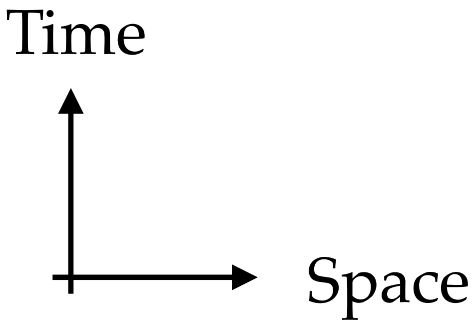


DB, Goodhew and Lee [2024]

# Timeless Cosmology

The physics before the hot Big Bang has been replaced by a kinematic flow on the spatial boundary:





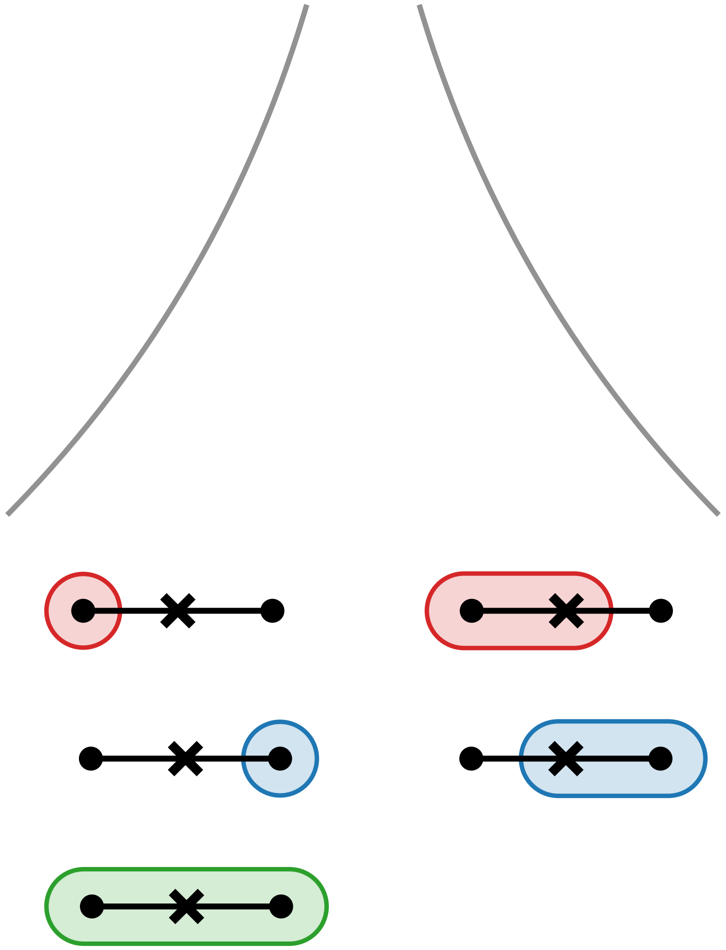
Is there a deeper mathematical structure underlying this flow?

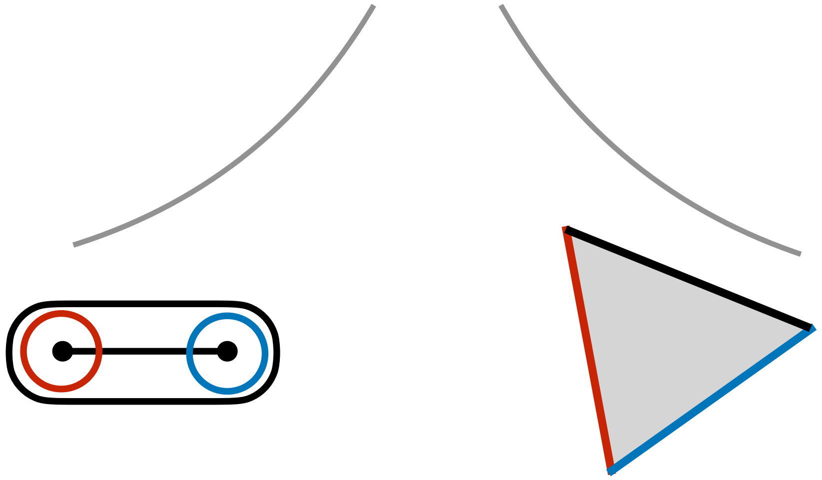
# Conclusions

# Summary

We developed a combinatorial description of cosmological correlators:

$$\psi(X_i, Y_j) = \int_0^\infty \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n \, (\omega_1 \cdots \omega_n)^\varepsilon \, \psi_{\text{flat}}(X_i + \omega_i, Y_j)$$





Simple rules determine the structure of all tree graphs and loop integrands.

# Open Problems

Many open problems remain:

- Why does this work?
- What is the geometric origin of the combinatorial rules?
- Is there a generalization to loop integrals?
- Is there a generalization to massive fields?

Please get in touch if you would like to discuss any of this!

Thanks For Your Attention

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