MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

Landau Analysis and Euler Stratifications

UNIVERSE+ Online Seminar
December 4, 2024

Simon Telen

What is Landau Analysis?

A scattering amplitude is a function of kinematic parameters

$$\mathscr{A}(z) = \sum_G I_G(z), \quad z \in Z \quad \textit{kinematic space}$$

The sum is over all admissible Feynman diagrams G

$$G \underset{\textit{Feynman rules}}{\longmapsto} I_G(z) = \int_{(\mathbb{R}^{1,D-1})^{L_G}} \frac{B(\ell, p)}{\prod_{e=1}^{E_G} (q_e(\ell, p)^2 - m_e{}^2)} \mathrm{d}^{D \cdot L_G} \ell$$

 # internal edges (points to E_G)

 # loops (points to L_G)

$I_G(z)$ is locally a multi-valued holomorphic function of $z = (p, m_e) \in Z$

Goal: determine the loci of poles and branch points of $\mathscr{A}(z)$

"Singularities"

… or of $I_G(z)$

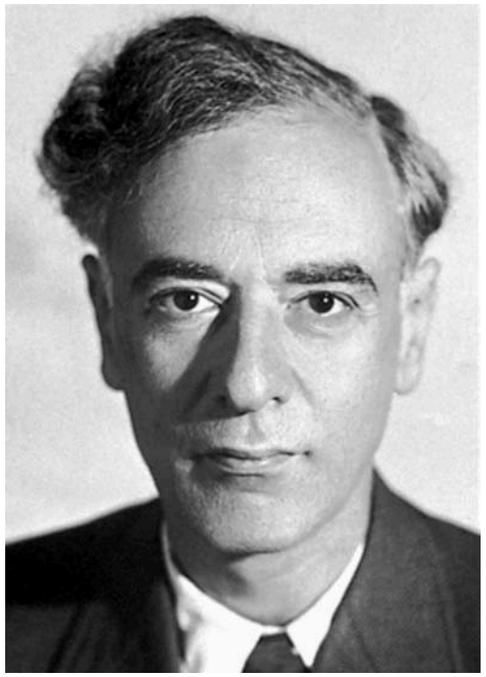
What is Landau Analysis?

SOVIET PHYSICS JETP VOLUME 37(10), NUMBER 1 JANUARY, 1960

ON THE ANALYTIC PROPERTIES OF VERTEX PARTS IN QUANTUM FIELD THEORY

L. D. LANDAU

A general method is developed, on the basis of the diagram technique, for finding the singularities of quantities involved in quantum field theory.



source: Wikipedia

An arbitrary diagram represents a certain integral

$$\int \frac{Bd^4kd^4l\ldots}{A_1A_2A_3\ldots}, \tag{1}$$

where

$$A_i = m_i^2 - q_i^2, \tag{2}$$

q_i is a certain four-momentum, corresponding to a given line in the diagram; m_i is the mass of the corresponding particle and B is a certain polynomial in the vectors q_i. According to the well-known method of Feynman, we can write

$$\frac{1}{A_1A_2\ldots A_n} = (n-1)!\int_0^1\ldots\int_0^1 \frac{d\alpha_1\,d\alpha_2\ldots d\alpha_n\cdot\delta(\alpha_1+\alpha_2+\ldots+\alpha_n-1)}{(\alpha_1A_1+\alpha_2A_2+\ldots+\alpha_nA_n)^n}. \tag{3}$$

⋮

Thus the singularity of the vertex part can be obtained by solving simultaneously the conditions $A_i = 0$ (or $\alpha_i = 0$) under the subsidiary conditions

$$\sum_i \alpha_i \frac{\partial A_i}{\partial k} = \sum_i \alpha_i \frac{\partial A_i}{\partial l} = \ldots = 0. \tag{7}$$

Pham, Thom and Whitney

APPLICATIONS OF AN ISOTOPY THEOREM

DIMITRI FOTIADI, MARCEL FROISSART, JEAN LASCOUX and FRÉDÉRIC PHAM

(*Received* 28 *June*, 1964)

INTRODUCTION

OUR ORIGINAL AIM was the study of the analyticity of a function defined by an integral:

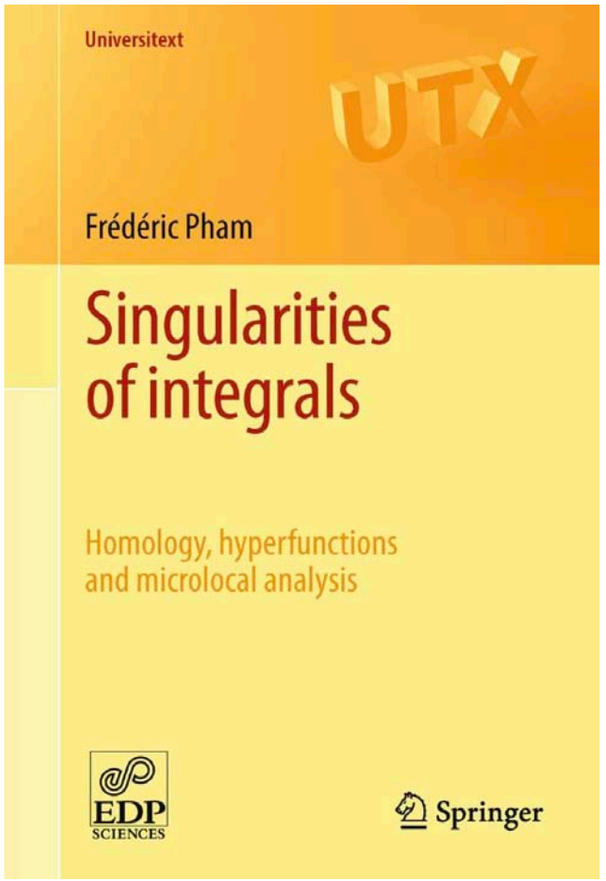
$$I(t) = \int_{\Gamma} \varphi_t(x) \qquad t \in T, \qquad x \in X$$

where T and X are complex analytic manifolds, Γ is a compact cycle in X, $\varphi_t(x)$ is a differential form which is closed in X and depends analytically on the parameter t.

Pham, Thom and Whitney

$$G \longmapsto I_G(z) = \int_{(\mathbb{R}^{1,D-1})^{L_G}} \frac{B(\ell, p)}{\prod_{e=1}^{E_G} (q_e(\ell, p)^2 - m_e{}^2)} \mathrm{d}^{D \cdot L_G} \ell$$

$$= \frac{\Gamma(\cdots)}{\Gamma(\cdots)\Gamma(\cdots)} \int_{\mathbb{R}_+^{E_G}} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_{E_G}}{(\mathcal{U}_G(x) + \mathcal{F}_G(x, z))^{\frac{D}{2}}}$$



1. Compute a Whitney stratification of $(d = E_G)$

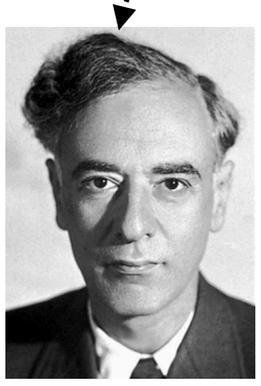
$$\mathcal{X} = V_{\mathbb{P}^d \times Z}(x_0 \cdots x_d (x_0 \mathcal{U}_G(x) + \mathcal{F}_G(x, z)))$$

2. On each stratum $S \subset \mathcal{X}$, determine the projection of loci on which $\pi : S \to Z$ is not submersive

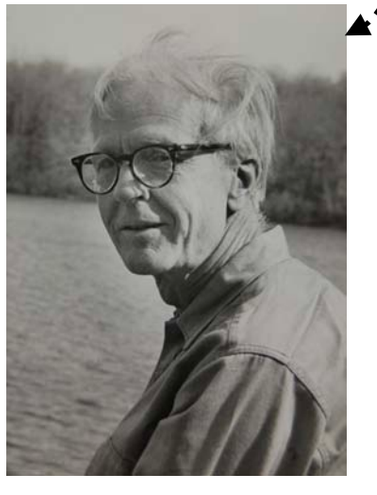
3. The Landau variety is the union of these loci.
It contains the singularities

application of Thom's isotopy lemma

Whitney stratifications and Landau varieties

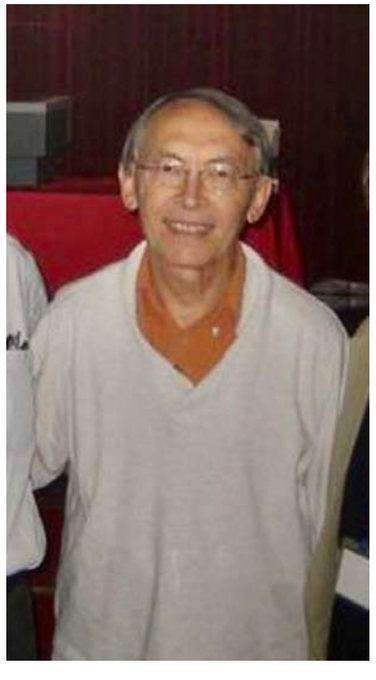


Landau



Whitney

Thom

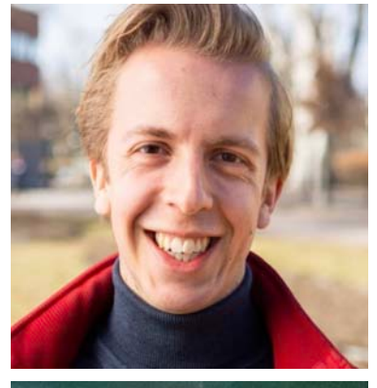


Pham

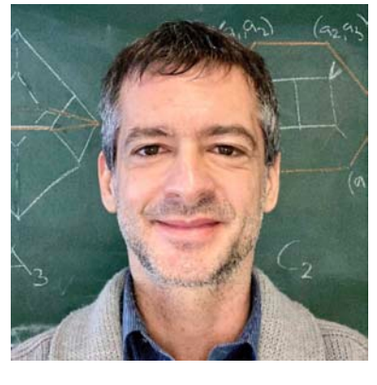
Brown

Panzer

Helmer



Tellander



Papathanasiou

Landau discriminants

Landau equations for mathematicians

(by Sebastian for Bernd & Simon)

Landau Discriminants, with Sebastian Mizera

JHEP

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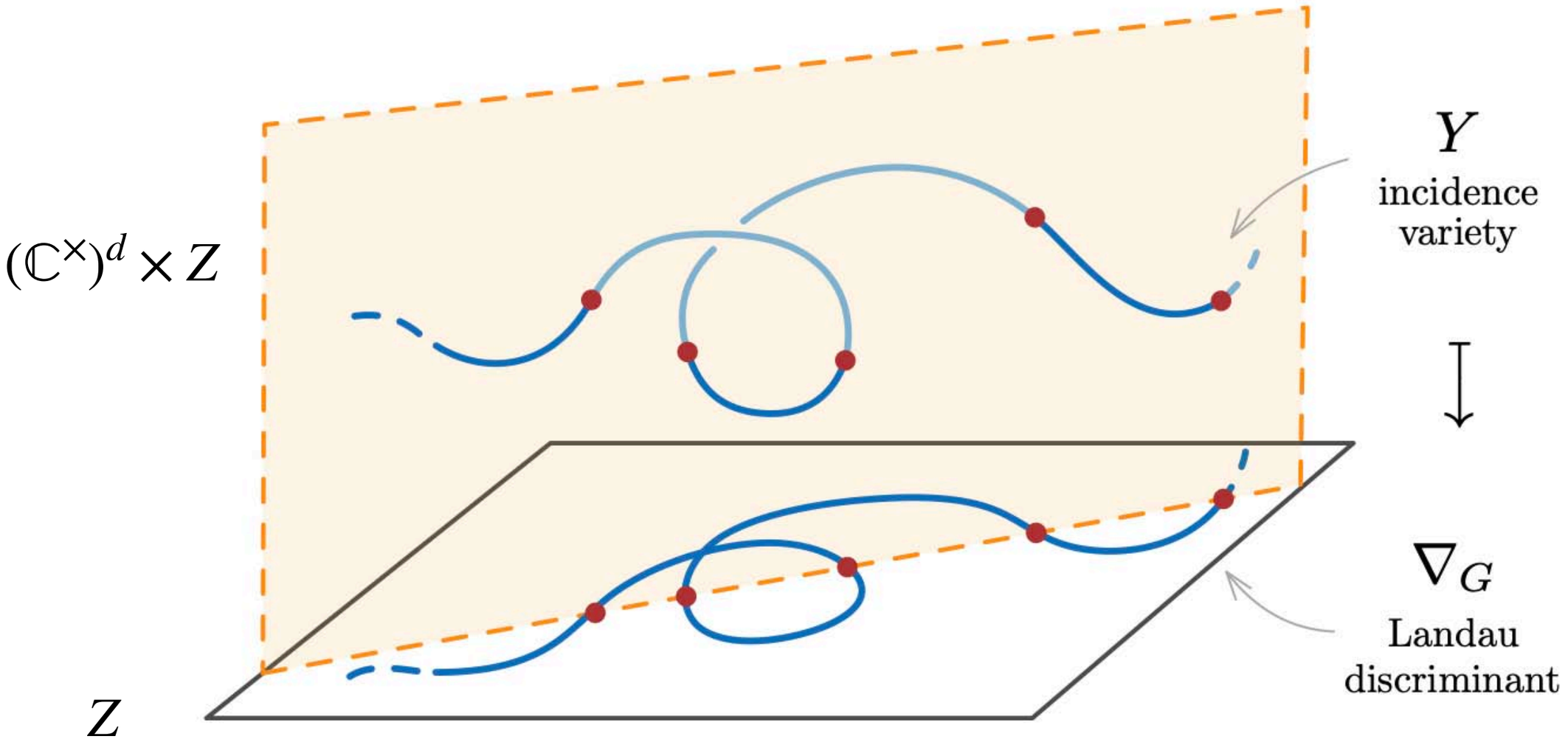
$$Y = \left\{ (x,z) \in (\mathbb{C}^\times)^d \times Z : \frac{\partial \mathscr{F}_G}{\partial x_1} = \cdots = \frac{\partial \mathscr{F}_G}{\partial x_d} = 0,\ \mathscr{U}_G \neq 0 \right\}$$

"Leading Landau equations"

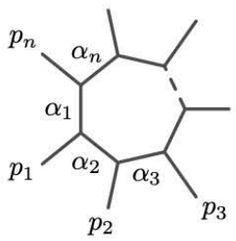
Theorem/Definition (Mizera, T.). For any graph G, the Landau discriminant $\overline{\pi_Z(Y)} \subset Z$ is an irreducible subvariety of codimension at least one.

Landau discriminants

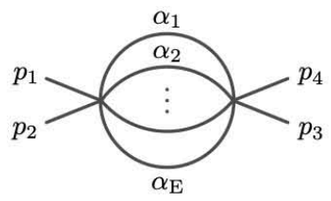
… via numerical algebraic geometry:



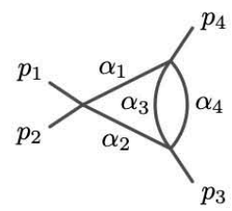
Landau discriminants



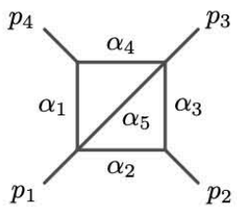
(a) One-loop n-gon diagram, $G = \mathtt{A}_n$ (Sec. 2.5)



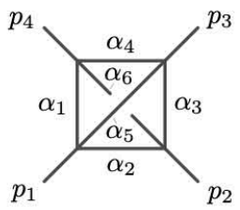
(b) Banana diagram with E edges, $G = \mathtt{B}_\mathrm{E}$ (Sec. 2.6, 4.4)



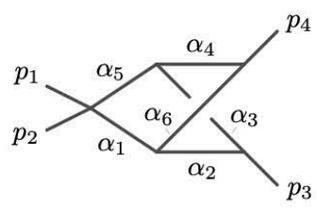
(c) Parachute diagram, $G = \mathtt{par}$ (Ex. 15, Thm. 2)



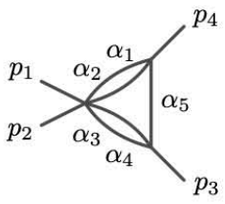
(d) Acnode diagram, $G = \mathtt{acn}$ (Ex. 10, Rk. 9, Thm. 2)



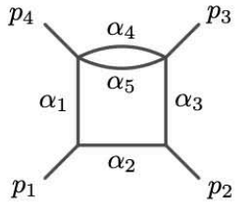
(e) Envelope diagram, $G = \mathtt{env}$ (Ex. 12, Sec. 3.4)



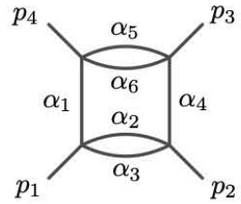
(f) Non-planar triangle-box diagram, $G = \mathtt{npltrb}$ (Thm. 2)



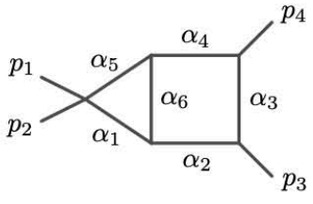
(g) Twice doubled-edge triangle diagram, $G = \mathtt{tdetri}$ (Thm. 2)



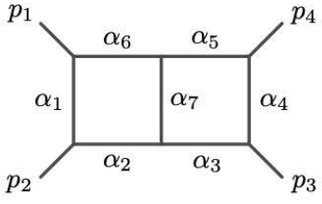
(h) Doubled-edge box diagram, $G = \mathtt{debox}$ (Thm. 2)



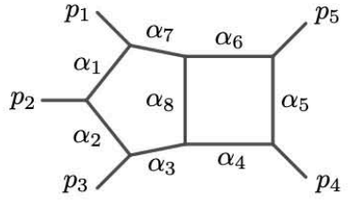
(i) Twice doubled-edge box diagram, $G = \mathtt{tdebox}$ (Thm. 2)



(j) Planar triangle-box diagram, $G = \mathtt{pltrb}$ (Sec. 3.3.1, Thm. 2)



(k) Double-box diagram, $G = \mathtt{dbox}$ (Thm. 2)

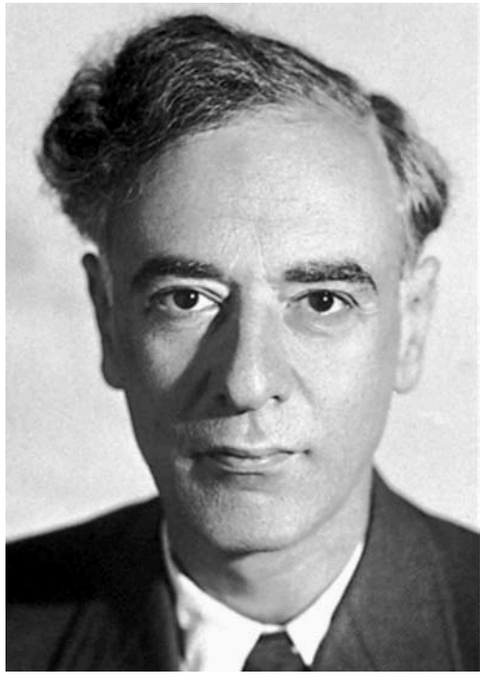


(l) Penta-box diagram, $G = \mathtt{pentb}$ (Ex. 13)

Diagram G	$\operatorname{codim} \nabla_G$	$\deg \nabla_G$	time (sec)
`par`	1	6	0.176
`acn`	1	16	0.489
`env`	1	114	13.1
`npltrb`	2	10	37.2
`tdetri`	1	12	1.04
`debox`	1	8	0.366
`tdebox`	1	16	10.5
`pltrb`	2	9	24.3
`dbox`	1	12	8.64
`pentb`	1	14	62.8

Diagram G	$\nabla_G(\mathcal{E})$	$t_{\mathtt{symb}}$ (sec)	$t_{\mathtt{num}}$ (sec)
`par`	$[1, 2]_1$	0.2	0.5
`acn`	$[4, 8]_1$	175306.0	1.4
`env`	$[8, 8, 8, 9, 12]_1$	$\times$	1226.1
`npltrb`	$[1, 1]_1$	1.9	4.0
`tdetri`	$[2]_1, [1]_2$	8.1	1.2
`debox`	$[3]_1, [1]_2$	7.9	0.5
`tdebox`	$[2]_1, [1]_2$	1476.8	4.3
`pltrb`	$[1, 1]_2$	0.6	$\times$
`dbox`	$[2, 4]_1$	13634.2	4.5
`pentb`	$[12]_1$	$\times$	815.9

Relation to classical discriminants

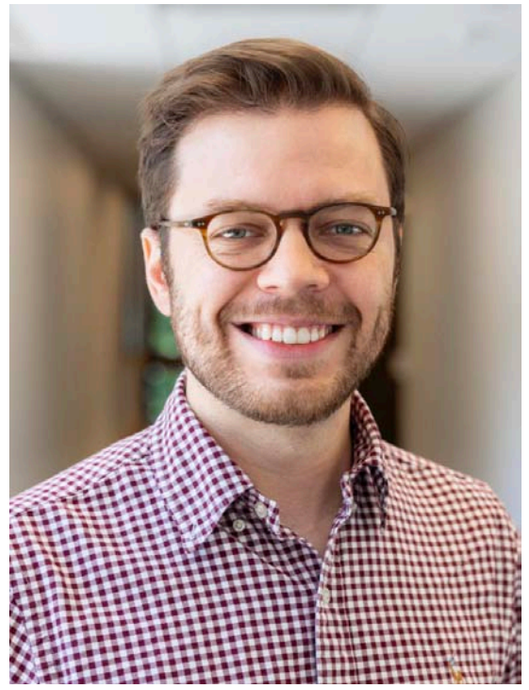


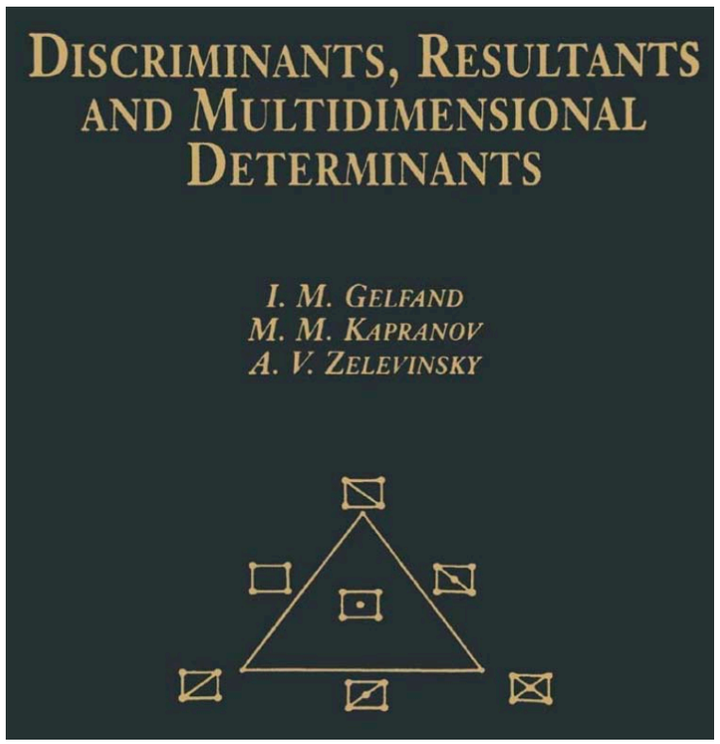
⋮

Thus the singularity of the vertex part can be obtained by solving simultaneously the conditions $A_i = 0$ (or $\alpha_i = 0$) under the subsidiary conditions

$$\sum_i \alpha_i \frac{\partial A_i}{\partial k} = \sum_i \alpha_i \frac{\partial A_i}{\partial l} = \ldots = 0. \qquad (7)$$

$$\frac{\partial \mathcal{F}_G}{\partial x_1} = \cdots = \frac{\partial \mathcal{F}_G}{\partial x_d} = 0$$





$$A = (a_0 \quad a_1 \quad \cdots \quad a_m) \in \mathbb{Z}^{(d+1)\times(m+1)}, \quad |a_i| = n$$

$$f_A(x, z) = z_0 x^{a_0} + z_1 x^{a_1} + \cdots + z_m x^{a_m}$$

$$Y_A = \left\{ (x, z) \in (\mathbb{C}^\times)^{d+1} \times Z : \frac{\partial f_A}{\partial x_0} = \cdots = \frac{\partial f_A}{\partial x_d} = 0 \right\}$$

The A-discriminant is $\nabla_A = \overline{\pi_Z(Y_A)}$

(it is irreducible, of codimension at least one)

Example: discriminants of quadrics

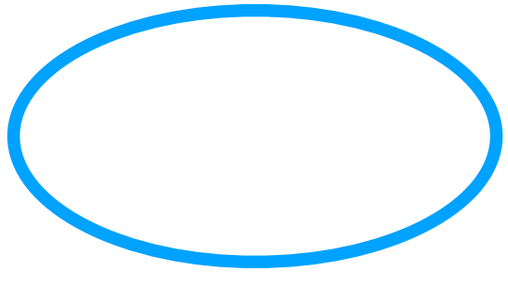
$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

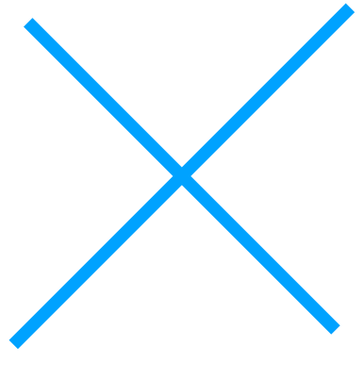
$$\Delta_A = z_1^2 - 4z_0z_2 = -\det \begin{pmatrix} 2z_0 & z_1 \\ z_1 & 2z_2 \end{pmatrix}$$

$$f_A(x,z) = z_0x_0^2 + z_1x_0x_1 + z_2x_1^2$$

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} \qquad M(z) = \begin{pmatrix} 2z_0 & z_1 & z_2 \\ z_1 & 2z_3 & z_4 \\ z_2 & z_4 & 2z_5 \end{pmatrix}, \quad \Delta_A = \det M(z)$$

$$f_A(x,z) = z_0x_0^2 + z_1x_0x_1 + z_2x_0x_2 + z_3x_1^2 + z_4x_1x_2 + z_5x_2^2$$





Euler integrals

$$\frac{\Gamma(\cdots)}{\Gamma(\cdots)\Gamma(\cdots)} \int_{\mathbb{R}_+^{E_G}} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_{E_G}}{(\mathcal{U}_G(x) + \mathcal{F}_G(x,z))^{\frac{D}{2}}}$$

$d = E_G$

$\nu \in \mathbb{C}^{d+1}$, generic

Γ "twisted cycle"

$$\int_\Gamma \frac{x_1^{\nu_1} \cdots x_d^{\nu_d}}{\tilde{f}(x,z)^{\nu_0}} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_d}{x_d}$$

Four lectures on Euler integrals

Saiei-Jaeyeong Matsubara-Heo, Sebastian Mizera, Simon Telen

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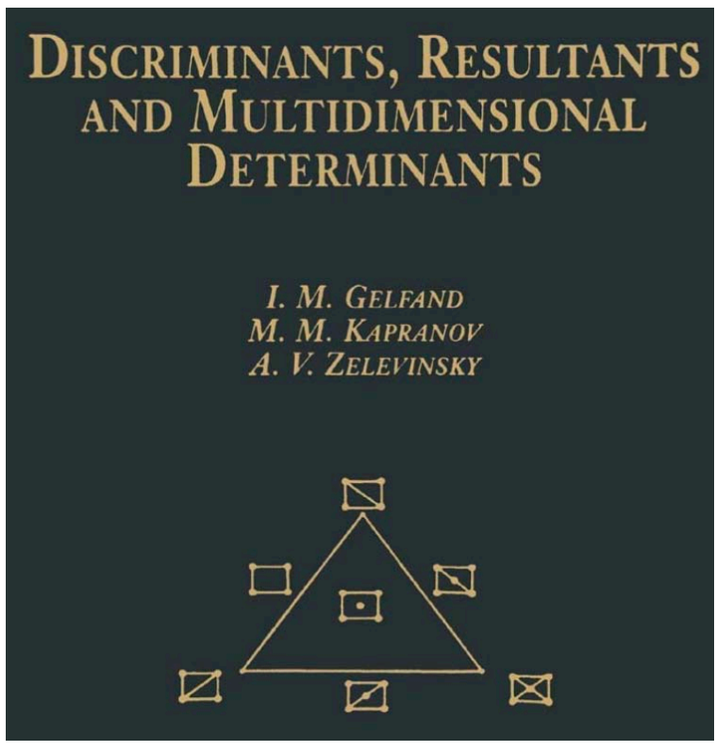


$$\int_\Gamma \frac{x_1^{\nu_1} \cdots x_d^{\nu_d}}{\tilde{f}_A(x,z)^{\nu_0}} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_d}{x_d}$$

$$\tilde{f}_A(x,z) = (z_0\, x^{a_0} + z_1\, x^{a_1} + \cdots + z_m\, x^{a_m})_{|x_0=1}$$

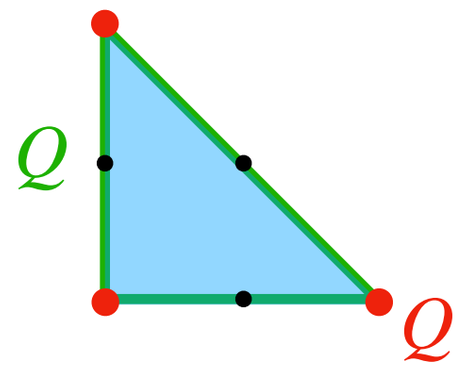
These *A-hypergeometric integrals* are solutions to a system of differential equations, whose singular locus has a nice (combinatorial) description

Principal A-determinants

… are built from A-discriminants

$$f_A(x,z) = z_0x_0^2 + z_1x_0x_1 + z_2x_0x_2 + z_3x_1^2 + z_4x_1x_2 + z_5x_2^2$$

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}, \qquad \operatorname{conv}(A) =$$



$$E_A = z_0 \cdot z_3 \cdot z_5 \cdot (z_1^2 - 4z_0z_3) \cdot (z_2^2 - 4z_0z_5) \cdot (z_4^2 - 4z_3z_5) \cdot \det M(z)$$

$$A \cap Q = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \qquad A \cap Q = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$M(z) = \begin{pmatrix} 2z_0 & z_1 & z_2 \\ z_1 & 2z_3 & z_4 \\ z_2 & z_4 & 2z_5 \end{pmatrix}$$

In general, the principal A-determinant is $E_A = \prod_{Q\,\text{face}} \Delta_{A\cap Q}^{e_Q}$

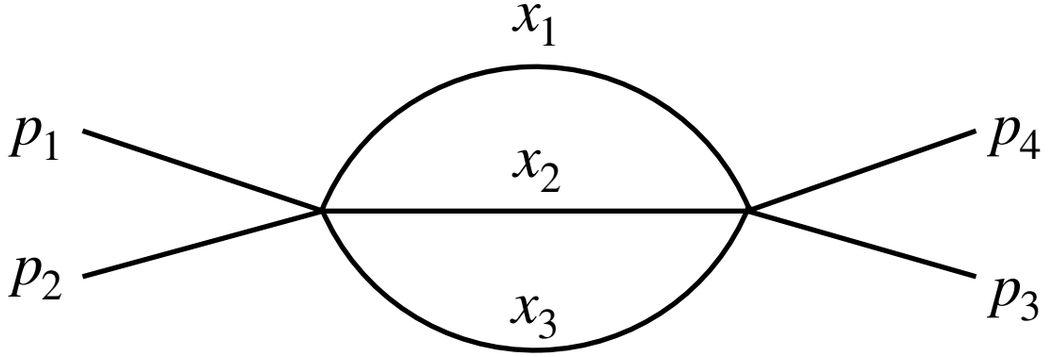
Theorem (GKZ). The zero locus of E_A contains the singularities of

$$\int_\Gamma \frac{x_1^{\nu_1} \cdots x_d^{\nu_d}}{\tilde{f}_A(x,z)^{\nu_0}} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_d}{x_d}$$

Principal Landau determinants

with Claudia Fevola and Sebastian Mizera

Goal: Pass from an A-hypergeometric integral to a parametric Feynman integral and specialize the principal A-determinant



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \end{pmatrix}$$

$$\int_\Gamma \Big[(1 - \sum_{i=1}^{3} m_i x_i)(x_1x_2 + x_1x_3 + x_2x_3) + s x_1x_2x_3\Big]^{-\nu_0} x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3} \frac{\mathrm{d}x_1}{x_1} \wedge \frac{\mathrm{d}x_2}{x_2} \wedge \frac{\mathrm{d}x_3}{x_3}$$

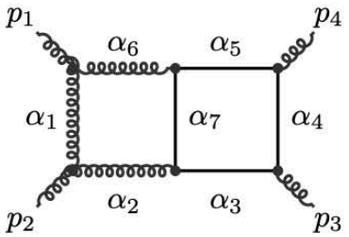
restrict to the kinematic space

$$(z_1, \ldots, z_{10}) = (1, 1, 1, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, -\mathsf{m}_3, -\mathsf{m}_3, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

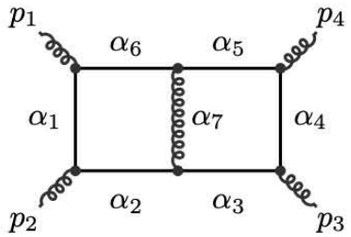
$$z_1\, x_1x_2 + z_2\, x_1x_3 + z_3\, x_2x_3 + z_4\, x_1^2x_2 + z_5\, x_1^2x_3 + z_6\, x_2^2x_3 + z_7\, x_1x_2^2 + z_8\, x_1x_3^2 + z_9\, x_2x_3^2 + z_{10}\, x_1x_2x_3$$

Principal Landau determinants

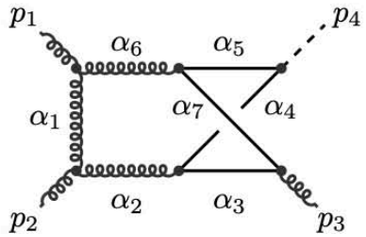
with Claudia Fevola and Sebastian Mizera



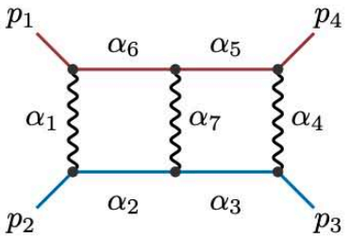
(a) Double-box with an inner massive loop, $G =$ `inner-dbox`



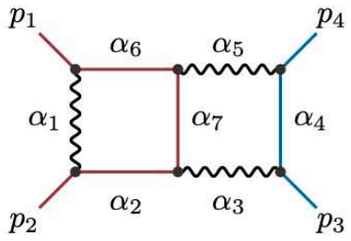
(b) Double-box with an outer massive loop, $G =$ `outer-dbox`



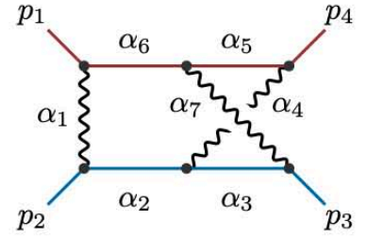
(c) Non-planar double-box for Higgs + jet production, $G =$ `Hj-npl-dbox`



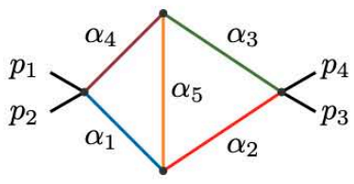
(d) Double-box for Bhabha scattering, $G =$ `Bhabha-dbox`



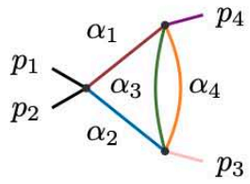
(e) Second double-box for Bhabha scattering, $G =$ `Bhabha2-dbox`



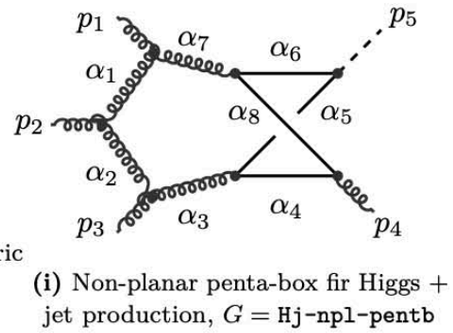
(f) Non-planar double-box for Bhabha scattering, $G =$ `Bhabha-npl-dbox`



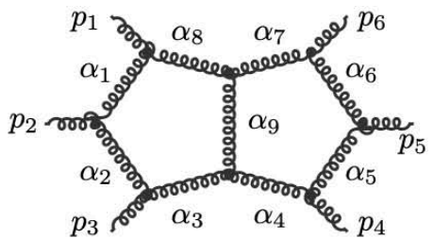
(g) Kite diagram with generic masses, $G =$ `kite`



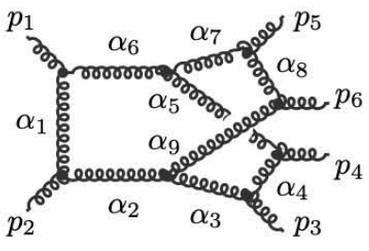
(h) Parachute diagram with generic masses, $G =$ `par`



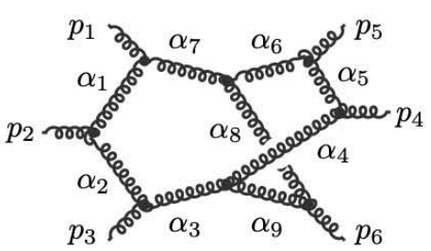
(i) Non-planar penta-box fir Higgs + jet production, $G =$ `Hj-npl-pentb`



(j) Massless planar double-pentagon, $G =$ `dpent`



(k) Massless non-planar double-pentagon, $G =$ `npl-dpent`



(l) Second massless non-planar double-pentagon, $G =$ `npl-dpent2`

PLD.jl available online

PLD.jl uses symbolic-numerical techniques from algebraic geometry to efficiently compute Landau components

Some components are missing!

More work on GKZ by de la Cruz, Dlapa, Helmer, Klausen, Papathanasiou, Tellander, …

Euler discriminants

Family of hypersurface complements:

$$\mathscr{X}_F = \{(x,z) \in \mathbb{P}^d \times Z : F(x,z) \neq 0\}, \quad \pi : \mathscr{X}_F \to Z$$

with $Z \subset \mathbb{P}^m$ irreducible, quasi-projective. Fibers are $\mathscr{X}_{F,z} = \pi^{-1}(z)$

Definition. The Euler discriminant ∇_χ of $\pi : \mathscr{X}_F \to Z$ is the closure of

$$\{z \in Z : \chi(\mathscr{X}_{F,z}) \text{ differs from its generic value}\}$$

Theorem (Esterov). If $F = x_0 \cdots x_d \, f_A(x,z)$, then the Euler discriminant of $\pi : \mathscr{X}_F \to Z$ is the vanishing locus of the principal A-determinant in $Z = \mathbb{P}^m$

$$\mathscr{X}_{F,z} = \{x \in (\mathbb{C}^\times)^d : \tilde{f}_A(x,z) = f_A(x,z)_{|x_0=1} \neq 0\}$$

Euler discriminants

Corollary. The singularities of $\int_\Gamma \frac{x_1^{\nu_1} \cdots x_d^{\nu_d}}{\tilde{f}_A(x,z)^{\nu_0}} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_d}{x_d}$ are contained in the Euler discriminant ∇_χ of $\mathscr{X}_F$, where $F = x_0 \cdots x_d f_A(x,z)$

Conjecture. The singularities of $\int_\Gamma \frac{x_1^{\nu_1} \cdots x_d^{\nu_d}}{\tilde{f}(x,z)^{\nu_0}} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_d}{x_d}$ are contained in the Euler discriminant ∇_χ of $\mathscr{X}_F$, where $F = x_0 \cdots x_d f(x,z)$

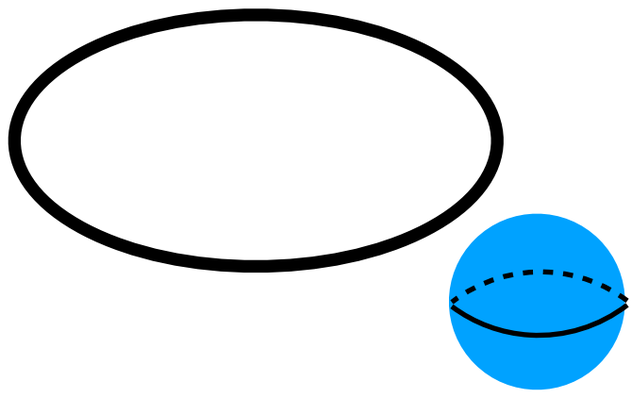
$$\mathscr{X}_{F,z} = \{x \in (\mathbb{C}^\times)^d : \tilde{f}(x,z) = f(x,z)_{|x_0=1} \neq 0\}$$

Ambitious goal. Compute Euler discriminants of families $\mathscr{X}_F$

More ambitious goal. Can we compute "strata" in Z on which $\chi(\mathscr{X}_{F,z})$ takes a prescribed (non-generic) value?

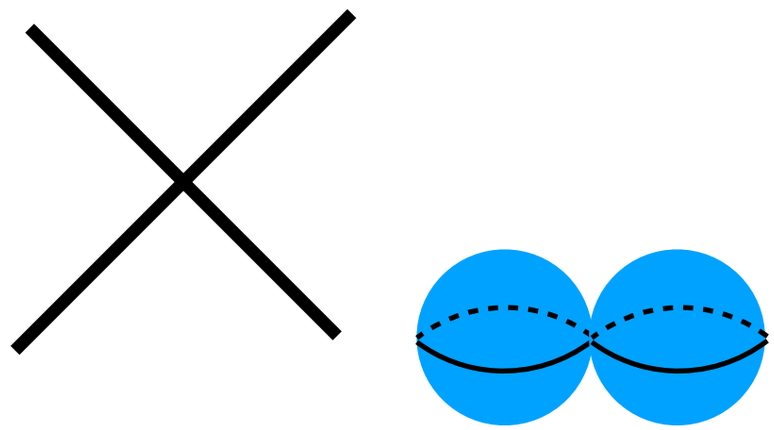
The Euler characteristic of plane conics

$$F(x_0, x_1, x_2, z) = z_0x_0^2 + z_1x_0x_1 + z_2x_0x_2 + z_3x_1^2 + z_4x_1x_2 + z_5x_2^2 = 0, \quad z \in \mathbb{P}^5_{\mathbb{C}}$$



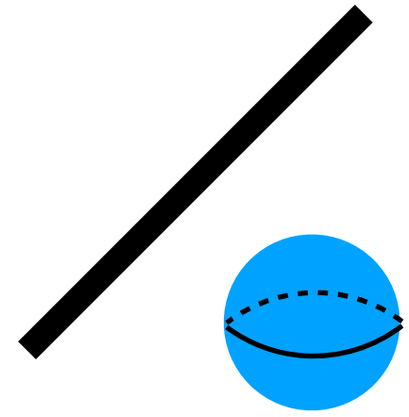
$$g = \frac{(n-1)(n-2)}{2} = 0$$

$$\chi = 2 - 2g = 2$$



$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

$$\chi(\mathbb{P}^1) + \chi(\mathbb{P}^1) - \chi(\mathrm{pt}) = 2 + 2 - 1 = 3$$

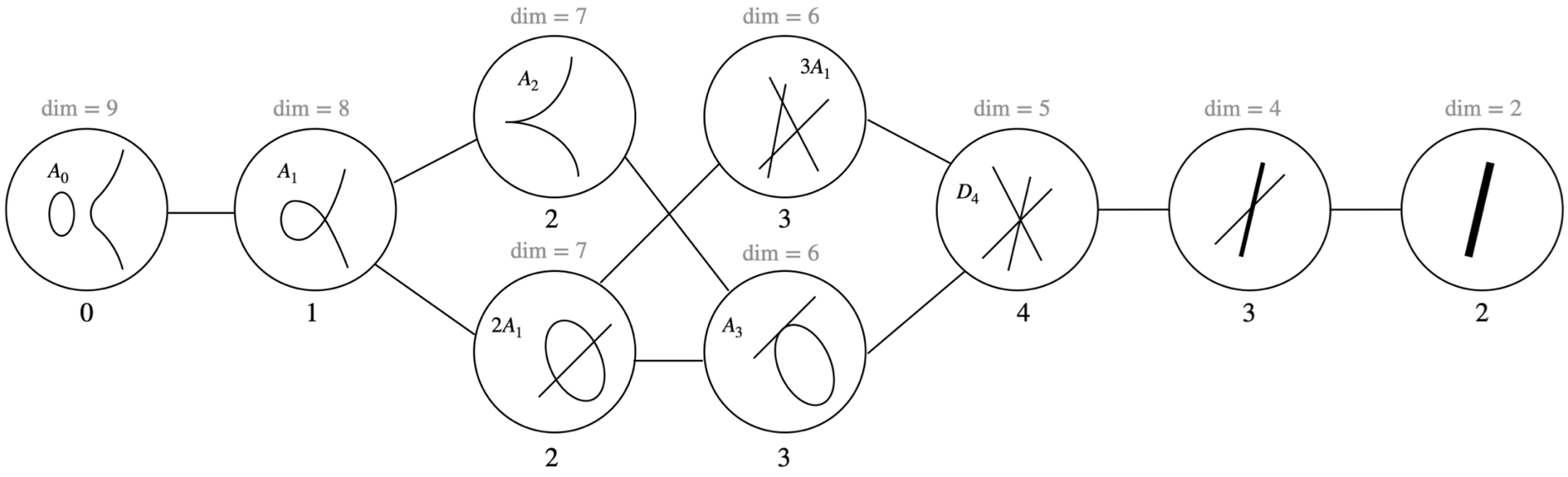


$$\chi(\mathbb{P}^1) = 2$$

$$\chi(V(F)) = 3 \iff \operatorname{rank} M(z) = \operatorname{rank} \begin{pmatrix} 2z_0 & z_1 & z_2 \\ z_1 & 2z_3 & z_4 \\ z_2 & z_4 & 2z_5 \end{pmatrix} = 2$$

Euler stratifications

paper with
Maximilian Wiesmann

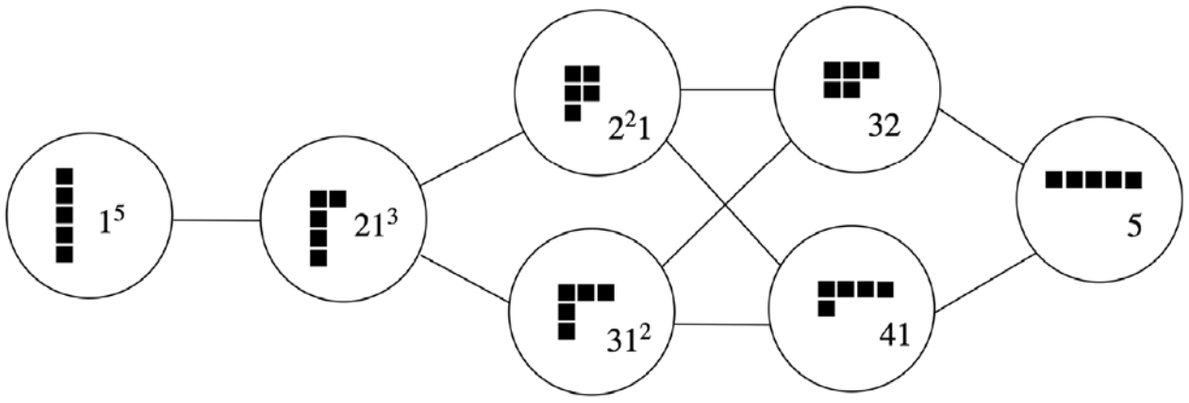


Definition 2.1. Let $\mathscr{X}, Z$ be quasi-projective varieties, with Z irreducible. Consider a surjective morphism $\pi : \mathscr{X} \to Z$ whose fibers have constant dimension. An *Euler stratification* of π is a finite set $\mathscr{S}$ of quasi-projective subvarieties of Z such that

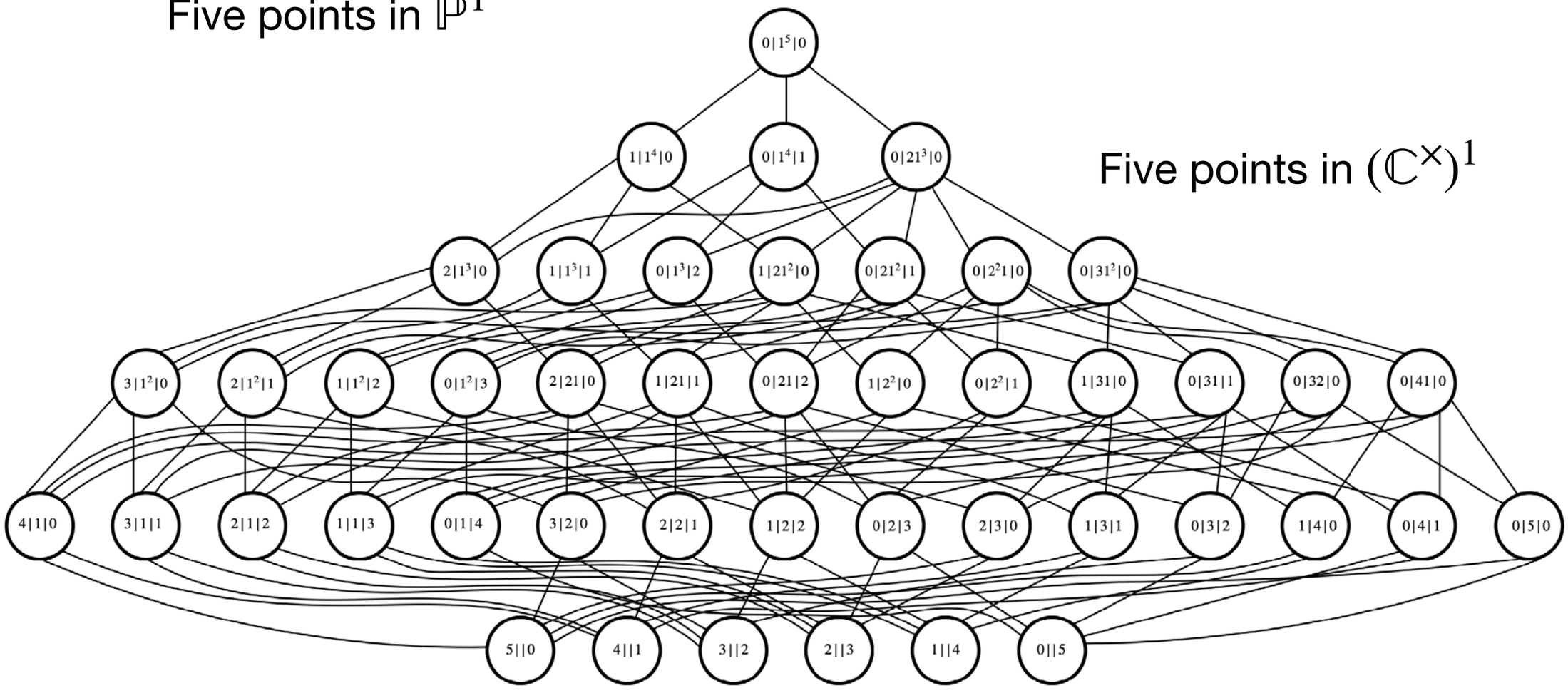
1. when $S \neq S' \in \mathscr{S}$, then $S \cap S' = \emptyset$, and $\bigsqcup_{S \in \mathscr{S}} S = Z$,
2. for each stratum $S \in \mathscr{S}$, the closure $\overline{S}$ is a union of strata: $\overline{S} = \bigsqcup_{S' \subseteq \overline{S}} S'$,
3. the Euler characteristic of the fiber $\chi(\pi^{-1}(z))$ is constant for $z \in S$.

The set $\mathscr{S}$ is partially ordered as follows: $S \preceq S'$ if and only if $\overline{S} \subseteq \overline{S'}$.

Hypersurfaces in the torus



Five points in $\mathbb{P}^1$



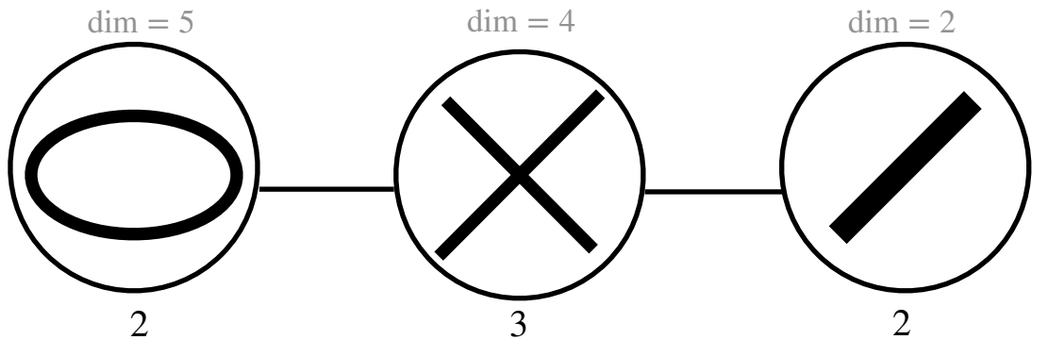
Five points in $(\mathbb{C}^\times)^1$

Semicontinuity + computation

$$\mathscr{X}_F = \{(x,z) \in \mathbb{P}^d \times Z : F(x,z) = x_0 x_1 \cdots x_d \cdot f(x,z) \neq 0\}$$

Theorem. $Z_{\leq k} = \{z \in Z : (-1)^d \cdot \chi(\mathscr{X}_{F,z}) \leq k\}$ is closed in Z

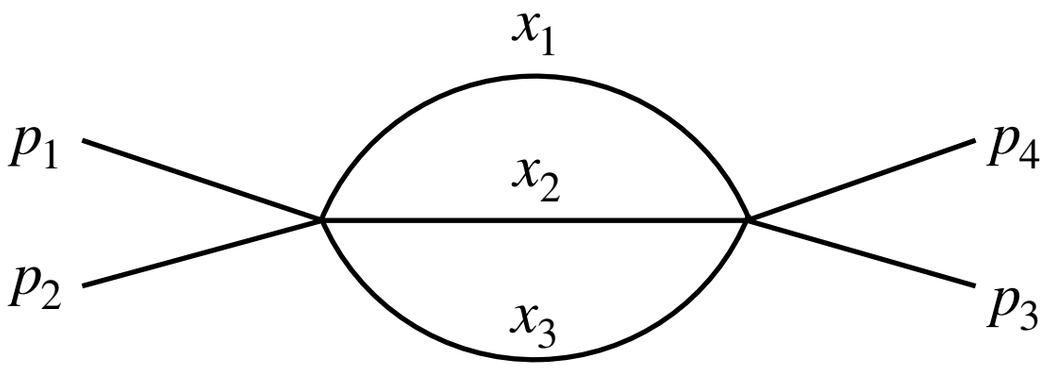
Not true in $\mathbb{P}^d$:



$$\mathscr{Y}_\nu^\circ = \left\{ (x,z) \in \mathscr{X}_F : \frac{\nu_i}{x_i} + \nu_0 \frac{\partial_i f}{f}(x,z) = 0,\ i = 1,\ldots,d \right\} \qquad \mathscr{Y}_\nu = \overline{\mathscr{Y}_\nu^\circ} \subset \mathbb{P}^d \times Z$$

Proposition. For generic ν, the projection $\pi_Z(\mathscr{Y}_\nu \cap V(F))$ is of codimension ≥ 1 in Z. It contains (is) the Euler discriminant of $\mathscr{X}_F$.

Based on a Theorem by June Huh, leads to symbolic algorithm



MATHREPO
MATHEMATICAL RESEARCH-DATA REPOSITORY

$$\int_\Gamma \left[(1-\sum_{i=1}^{3} m_i x_i)(x_1x_2+x_1x_3+x_2x_3)+sx_1x_2x_3\right]^{-\nu_0} x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}\,\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_2}{x_2}\wedge\frac{\mathrm{d}x_3}{x_3}$$

$$\begin{aligned}\Delta_\chi \;=\; & m_1\cdot m_2\cdot m_3\cdot s\cdot(m_1^4-4m_1^3m_2-4m_1^3m_3-4m_1^3s+6m_1^2m_2^2+4m_1^2m_2m_3\\ & +4m_1^2m_2s+6m_1^2m_3^2+4m_1^2m_3s+6m_1^2s^2-4m_1m_2^3+4m_1m_2^2m_3+4m_1m_2^2s\\ & +4m_1m_2m_3^2-40m_1m_2m_3s+4m_1m_2s^2-4m_1m_3^3+4m_1m_3^2s+4m_1m_3s^2\\ & -4m_1s^3+m_2^4-4m_2^3m_3-4m_2^3s+6m_2^2m_3^2+4m_2^2m_3s+6m_2^2s^2-4m_2m_3^3\\ & +4m_2m_3^2s+4m_2m_3s^2-4m_2s^3+m_3^4-4m_3^3s+6m_3^2s^2-4m_3s^3+s^4).\end{aligned}$$

```
d = 3
Rz, z = polynomial_ring(QQ, ["m_1";"m_2";"m_3";"s"])
Rx, x = polynomial_ring(Rz, "x" => 1:d+1)
f = ((x[4]-sum([z[i]*x[i] for i = 1:3]))*(x[1]*x[2]+x[1]*x[3]+x[2]*x[3]) +
z[4]*x[1]*x[2]*x[3])
Iz = ideal([Rz(0)])
get_euler_discriminant_veryaff(f, Rz, Rx, Iz)
```

D-module whose singular locus is ∇_χ?

Euler VS Whitney?

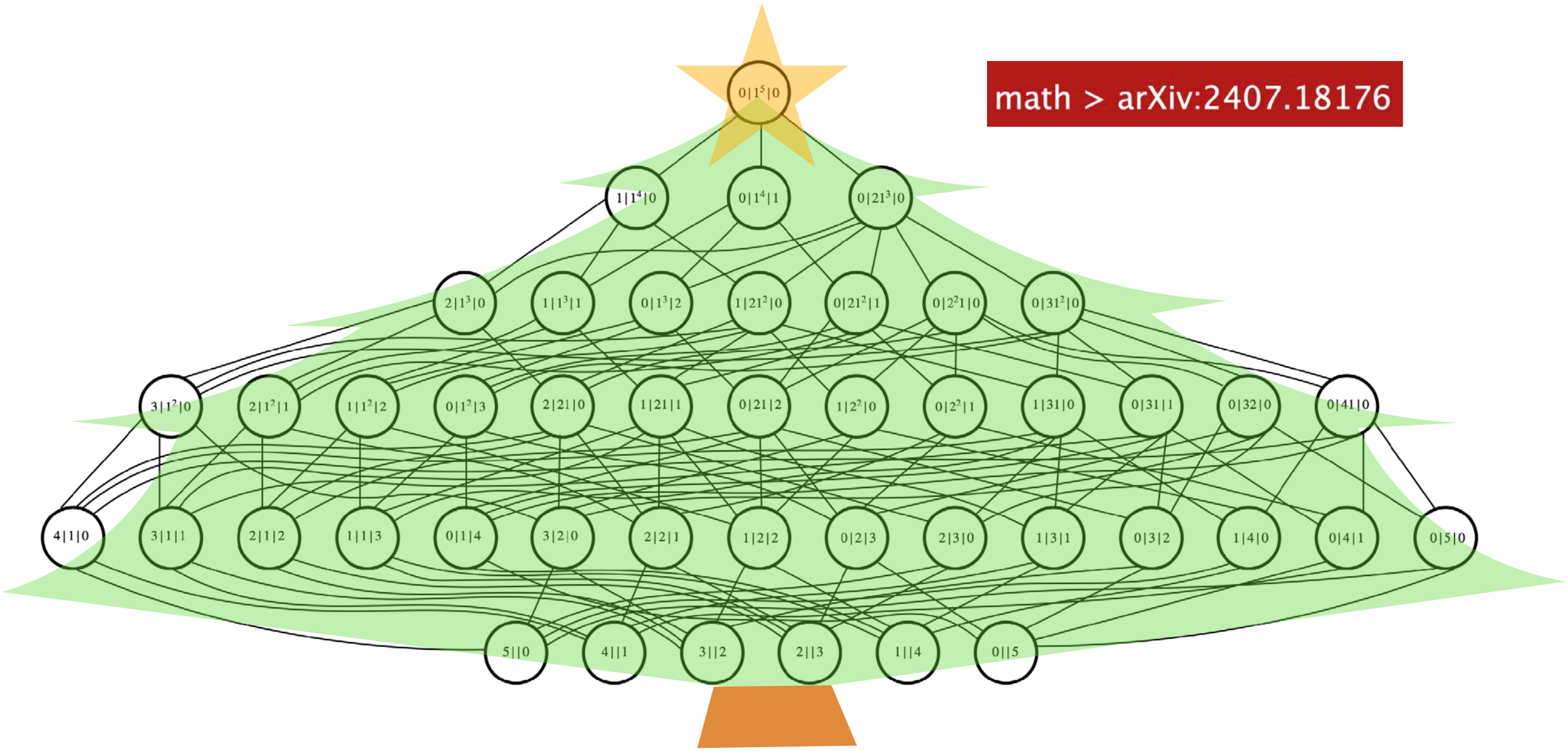
Euler stratifications in higher codimension?

Numerical Euler/Whitney stratifications?

Strata of minimal Euler characteristic?

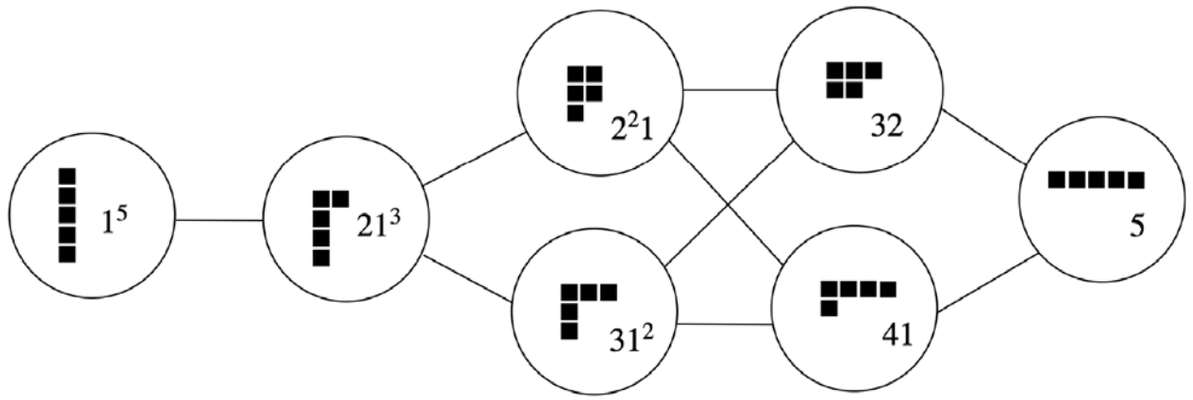
…

Thank you!



math > arXiv:2407.18176

Points on the line



Parametrizing these strata is easy:

$$(\mathbb{P}^1)^k \dashrightarrow S_\lambda, \quad ((a_1 : b_1), \ldots, (a_k : b_k)) \longmapsto (a_1x_0 + b_1x_1)^{\lambda_1} \cdots (a_kx_0 + b_kx_1)^{\lambda_k}$$

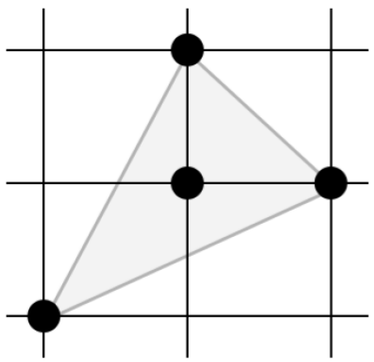
Computing implicit equations is hard and interesting:

Conjecture: the ideal of the stratum 2^21^{n-4} is generated by $3n-5$ forms of degree $2n-5$

λ	generators of $I(\nabla_\lambda)$	λ	generators of $I(\nabla_\lambda)$
(8)	2^{28}		
(7,1)	2^{15}	(4,4)	3^{74}
(6,1,1)	$2^6,\ 3^1$	(3,1,1,1,1,1)	$6^1,\ 7^6,\ 8^6$
(6,2)	$2^6,\ 3^1,\ 4^{46}$	(3,2,1,1,1)	$6^1,\ 7^6,\ 8^6,\ 10^{46}$
(5,1,1,1)	$2^1,\ 3^6,\ 4^6$	(3,2,2,1)	$6^1,\ 7^6,\ 8^{546}$
(5,2,1)	$2^1,\ 3^6,\ 4^6,\ 6^{54}$	(3,3,1,1)	5^{84}
(5,3)	$2^1,\ 3^{19},\ 4^{64}$	(3,3,2)	$4^1,\ 5^{166},\ 6^{100}$
(4,1,1,1,1)	$4^5,\ 5^9,\ 6^1$	(2,1,1,1,1,1,1)	14^1
(4,2,1,1)	$4^6,\ 5^9,\ 6^1,\ 8^{54}$	(2,2,1,1,1,1)	11^{19}
(4,2,2)	$4^{15},\ 5^{221}$	(2,2,2,1,1)	8^{120}
(4,3,1)	4^{45}	(2,2,2,2)	5^{286}

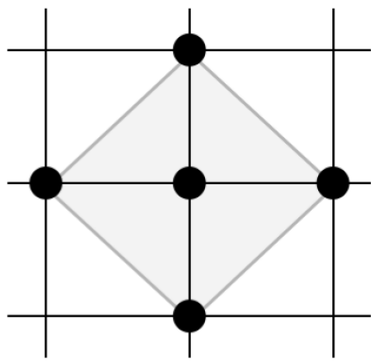
[33] H. Lee and B. Sturmfels. Duality of multiple root loci. *Journal of Algebra*, 446:499–526, 2016.

Toric Fano surfaces



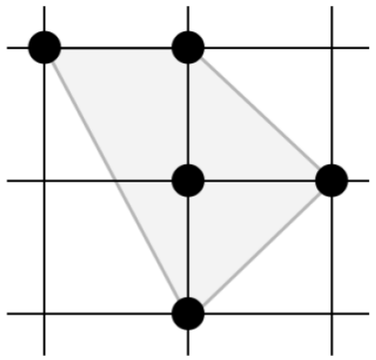
$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$(3,1,3)^1$
$(2,1,1)^3$
$(2,3,2)^1$
$(1,1,0)^6$



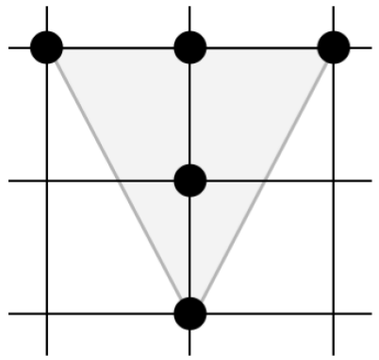
$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$(4,1,4)^1$
$(3,1,2)^4$
$(3,4,3)^1$
$(2,1,1)^4$
$(2,2,1)^4$
$(2,2,2)^1$
$(2,1,0)^2$
$(1,1,0)^8$
$(1,2,0)^2$



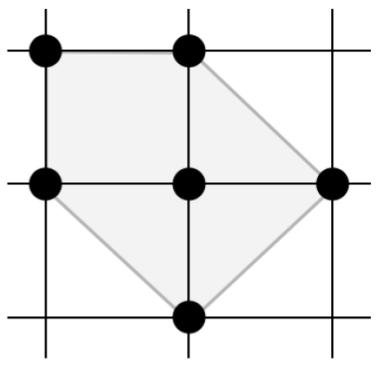
$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}$$

$(4,1,4)^1$
$(3,1,2)^3$
$(3,1,3)^1$
$(3,5,3)^1$
$(2,1,1)^6$
$(2,2,1)^3$
$(2,1,0)^1$
$(2,4,2)^1$
$(2,3,2)^1$
$(1,1,0)^9$
$(1,2,0)^1$



$$\begin{pmatrix} 3 & 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$(4,1,4)^1$
$(3,1,2)^3$
$(3,2,3)^1$
$(3,4,3)^1$
$(2,1,0)^2$
$(2,1,1)^4$
$(2,2,1)^3$
$(2,4,2)^1$
$(2,2,2)^1$
$(1,1,0)^8$
$(1,2,0)^2$



$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

$(5,1,5)^1$
$(4,1,4)^3$
$(4,1,3)^3$
$(4,8,4)^1$
$(3,1,2)^9$
$(3,5,3)^2$
$(3,1,3)^1$
$(3,2,2)^2$
$(3,3,2)^2$
$(3,12,3)^1$
$(3,3,3)^1$

$(3,4,3)^1$
$(2,1,1)^{11}$
$(2,2,1)^9$
$(2,4,2)^2$
$(2,3,2)^1$
$(2,3,1)^2$
$(2,5,2)^1$
$(2,1,0)^2$
$(2,2,2)^1$
$(1,1,0)^{13}$
$(1,2,0)^2$

$(\dim \overline{S}, \deg \overline{S}, \chi(\mathcal{X}^*_{f,z}))^e$

Maximum likelihood estimation

A statistical model for a discrete random variable with s states is a subset of the probability simplex of dimension $s-1$

Joint probability distribution of two binary random variables

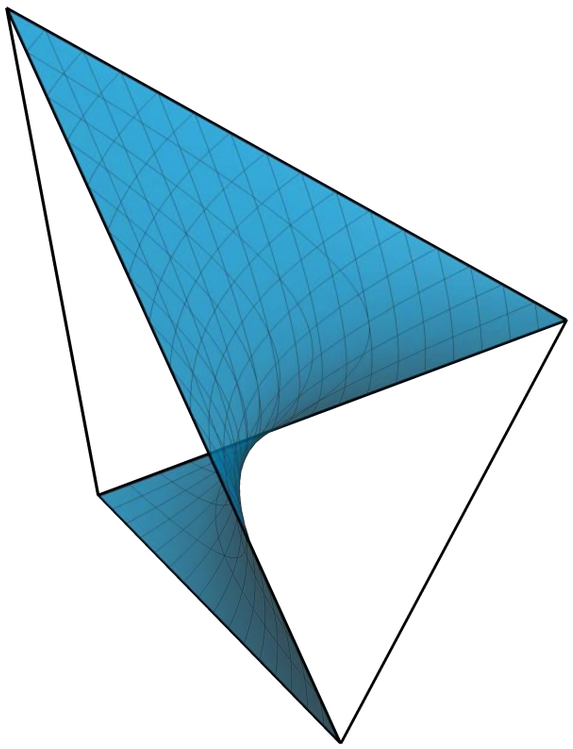
	LOTR	HP
Red	p_{00}	p_{01}
White	p_{10}	p_{11}

independence:

$$p_{\text{LOTR}} + p_{\text{HP}} = 1, \quad p_{\text{red}} + p_{\text{white}} = 1$$

$$p_{00} = p_{\text{LOTR}} \cdot p_{\text{red}}$$

$$= \begin{bmatrix} p_{\text{red}} \\ p_{\text{white}} \end{bmatrix} \cdot [p_{\text{LOTR}} \quad p_{\text{HP}}]$$



$$p_{00} = \frac{x_0y_0}{f}, \quad p_{01} = \frac{x_0y_1}{f}, \quad p_{10} = \frac{x_1y_0}{f}, \quad p_{11} = \frac{x_1y_1}{f}, \quad f = x_0y_0 + x_0y_1 + x_1y_0 + x_1y_1$$

This model is the intersection of the Segre quadric in $\mathbb{P}^3$ with the probability simplex $\mathbb{P}^3_{>0}$

Maximum likelihood estimation

A statistical model for a discrete random variable with s states is a subset of the probability simplex of dimension $s-1$

independence model of two binary random variables

$$p_{00}=\frac{x_0y_0}{f},\quad p_{01}=\frac{x_0y_1}{f},\quad p_{10}=\frac{x_1y_0}{f},\quad p_{11}=\frac{x_1y_1}{f},\quad f=x_0y_0+x_0y_1+x_1y_0+x_1y_1$$

suppose that, in an experiment, we observe state ij a total amount of u_{ij} times

MLE: infer $x=\frac{x_0}{x_1}$ and $y=\frac{y_0}{y_1}$ by maximizing $L_u = \log p_{00}^{u_{00}}p_{01}^{u_{01}}p_{10}^{u_{10}}p_{11}^{u_{11}}$

$$\frac{\partial L_u}{\partial x}=\frac{\partial L_u}{\partial y}=0 \rightarrow \boxed{\text{Homotopy Continuation.jl}} \rightarrow x=\frac{u_{00}+u_{01}}{u_{00}+u_{01}+u_{10}+u_{11}},\quad y=\frac{u_{00}+u_{10}}{u_{00}+u_{01}+u_{10}+u_{11}}$$

The number of complex solutions for generic data u is called the maximum likelihood degree of the model

The maximum likelihood degree

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