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UNIVERSE+ Online Seminar

Xavier Kervyn

“Structure and features of high-energy string amplitudes”

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Structure and features of high-energy string amplitudes

UNIVERSE+ online seminar, 14 October 2026

Xavier Kervyn
Max Planck Institute for Physics

Based on work in collaboration with Stephan Stieberger
[2604.07444]

Connection problem for exponential periods

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Exponential periods (à la KS)

- Consider integrals of the form $$I_\Gamma(\lambda) = \int_\Gamma e^{\lambda W(z)} \omega(z)$$

[Kontsevich, Soibelman '24]
'Holomorphic Floer Theory'

with: $\Gamma \subset X$ a path on a complex curve, ω a meromorphic 1-form, W a **multivalued** potential, λ a large parameter

$\alpha = \mathrm{d}W$ sv'd and meromorphic 1-form, zeros $\mathscr{Z}(\alpha)$

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- As $\lambda \to \infty$, asymptotic expansion (saddle-point approximation)

$$I_\Gamma(\lambda) \sim \sum_{\sigma \in \mathscr{Z}(\alpha)} n_\sigma e^{\lambda W(s_\sigma)} \lambda^{-1/2} \sum_{k \geq 0} c_{k,\sigma} \lambda^{-k}$$

local: Taylor expansion around s_σ, **asymptotic**

global: depend on Γ and region of parameter space, **discontinuous** (Stokes phenomenon)

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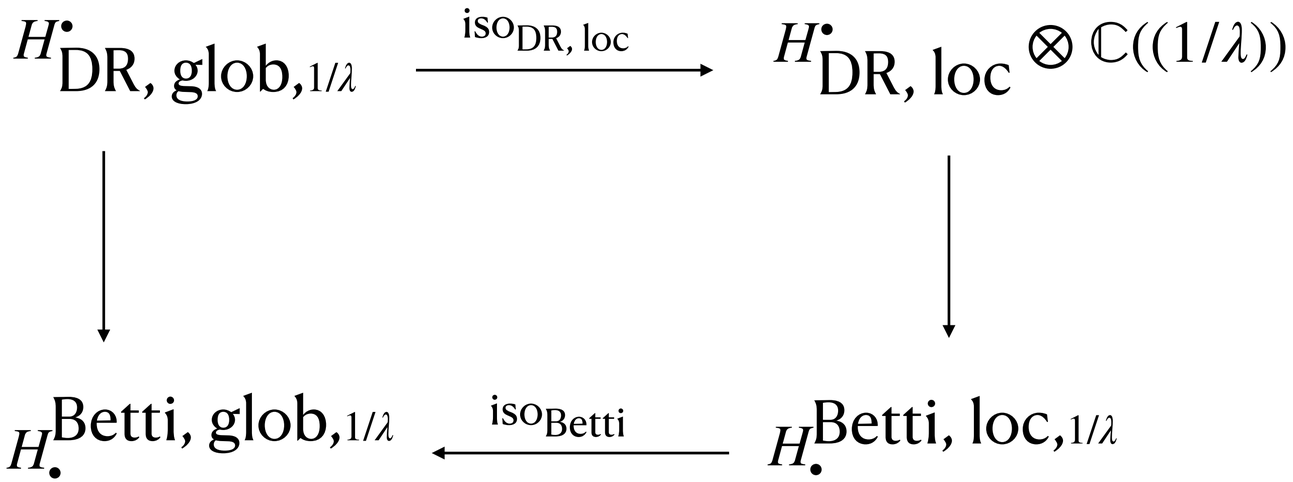
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- **Connection problem**: given the expansion in one region, find n_σ <u>everywhere</u>.

  $\Rightarrow$ 2 POVs: geometry of cycles (KS), <u>resurgence & difference equations</u> [XK, Stieberger '26]

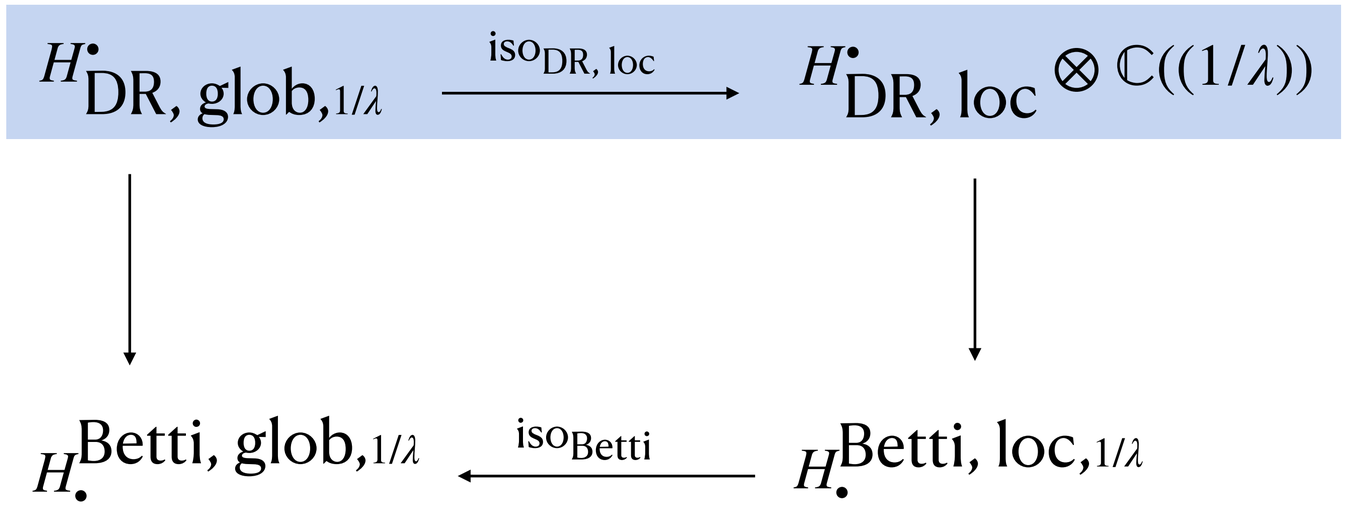
Global and local twisted (co)homology

[Kontsevich, Soibelman '24] [Witten '11]



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Global form determined globally by its Taylor data at the critical points

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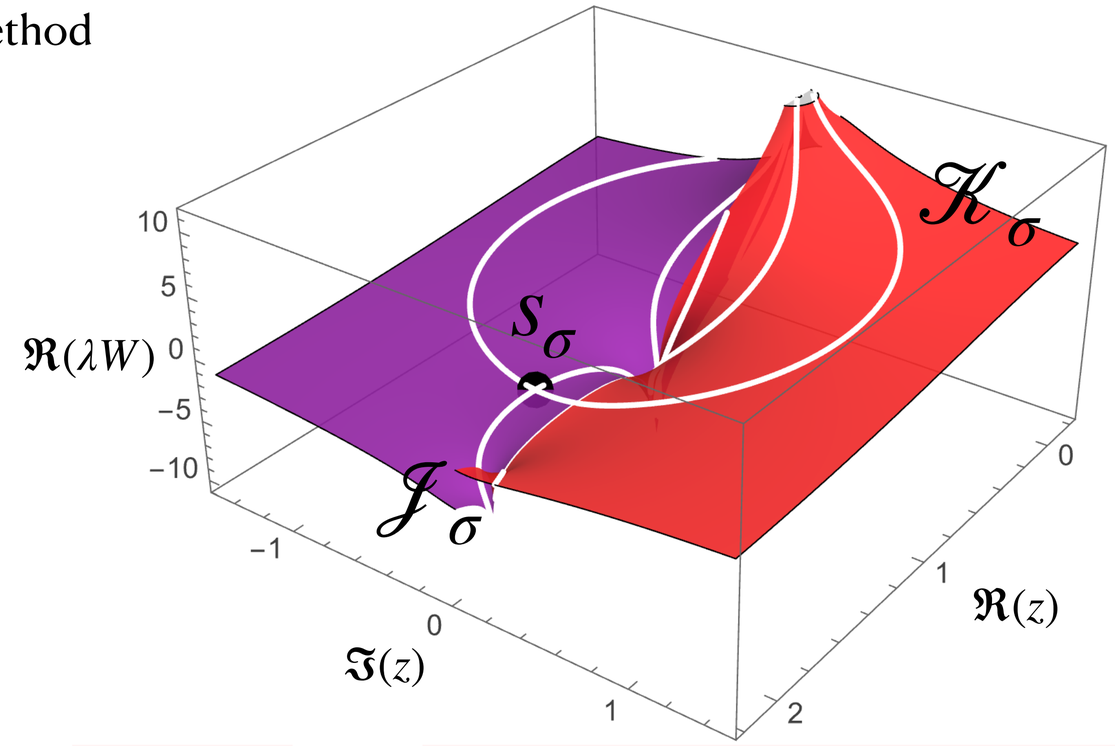
$$H^\bullet_{\mathrm{DR,\,glob},1/\lambda} \xrightarrow{\mathrm{iso}_{\mathrm{DR,\,loc}}} H^\bullet_{\mathrm{DR,\,loc}} \otimes \mathbb{C}((1/\lambda))$$

period pairing
$I_\Gamma(\lambda) = \langle \Gamma \otimes e^{\lambda W}, \omega \rangle$

Laplace's method
$I_{\mathscr{J}_\sigma}(\lambda)$

$$H^\bullet_{\mathrm{Betti,\,glob},1/\lambda} \xleftarrow{\mathrm{iso}_{\mathrm{Betti}}} H^\bullet_{\mathrm{Betti,\,loc},1/\lambda}$$

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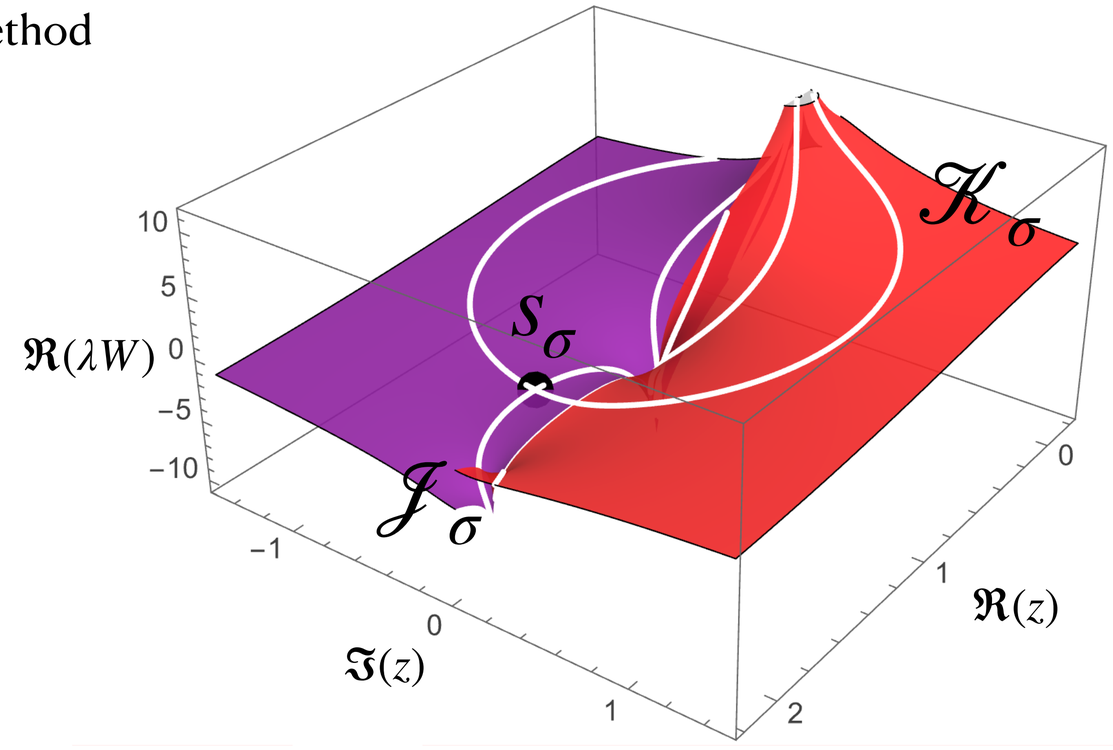
$$\Gamma \simeq \sum_{\sigma+\mathrm{lifts}} n_\sigma \mathscr{J}_\sigma \qquad n_\sigma = \langle \Gamma, \mathscr{K}_\sigma \rangle$$

$$\mathscr{J}_\sigma \mapsto \mathscr{J}_\sigma \pm n_{\sigma\sigma'} \mathscr{J}_{\sigma'} \quad (\&\ \text{infinite lifts})$$

$$Z(\gamma) = \int_\gamma \mathrm{d}W \qquad n_\sigma\text{: Novikov tube } \sum_\gamma c_\gamma e^{\lambda Z(\gamma)} \sim \text{Witten's } i\epsilon \quad [\text{Witten '13}]$$

$\Rightarrow$ global Betti = local Betti $\otimes$ coefs n_σ

Global form determined globally by its Taylor data at the critical points



Tree-level string amplitudes are KS periods

- At n points: generalized hypergeometric functions
  [Mafra, Schlotterer, Stieberger '11] [Aomoto, Kita '11]

$$X = \mathscr{M}_{0,n} \qquad W = \sum_{i<j} s_{ij} \log(z_i - z_j)$$

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- $\lambda = \alpha' \to 0$: field theory, MZVs

  [Brown '09] [Stieberger '11]
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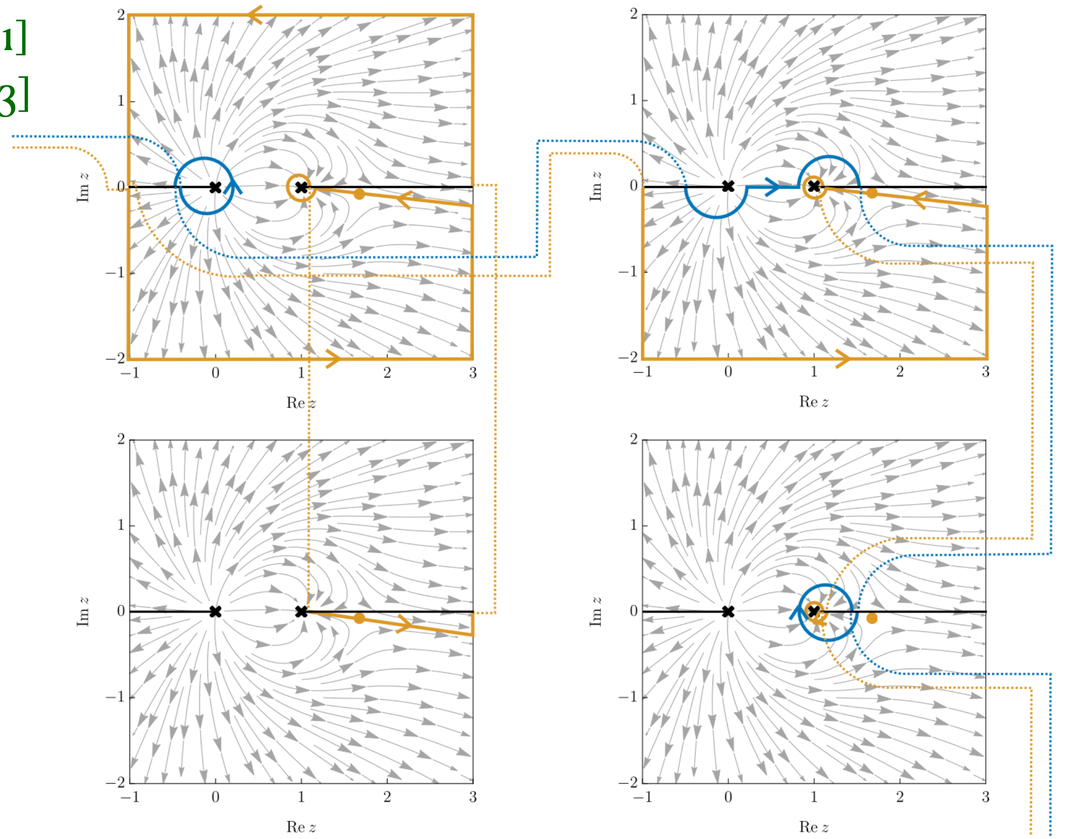
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- $\alpha' \to \infty$: high-energy limit (QG), $c_{k,\sigma} = ?$
  damping, contour deformations, numerics for $\alpha' < \infty$

[Gross, Mende '87 '88] [Gross, Manes '89] $g = 0$: [Eberhardt, Mizera '24] [Yoda '24]
$g = 1$: [Baccianti, Eberhardt, Mizera '25 '26]

Complex saddles (monodromy): $W \mapsto W \pm 2\pi i k s_{ij}$

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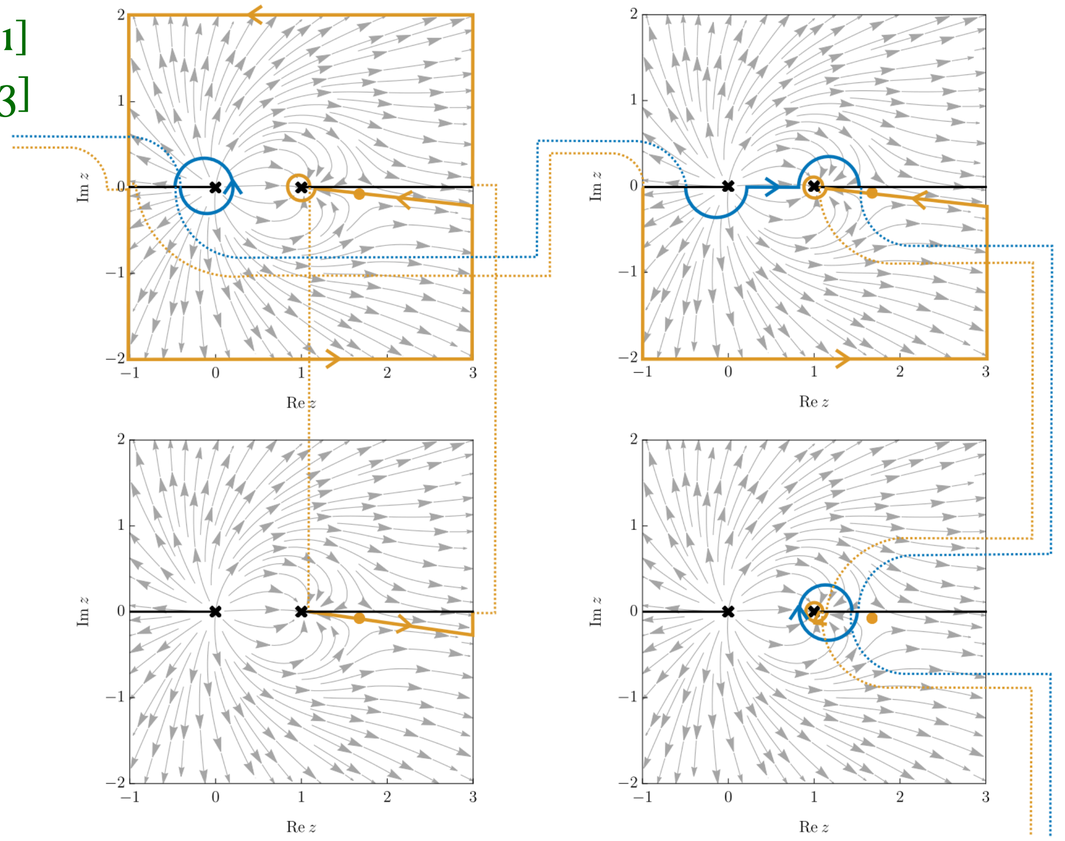
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Constructing thimbles is hard! $\Rightarrow$ **Resurgence**

[Aomoto '87] for $n = 5$

$$X = \mathscr{M}_{0,n} \qquad W = \sum_{i<j} s_{ij} \log(z_i - z_j)$$



Resurgence of Γ

$$\ln\Gamma(z) \sim \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \underbrace{\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}z^{1-2k}}_{S_0(z)} \qquad B_{2k} = -2k\,\zeta(1-2k) \in \mathbb{Q}$$

$$\uparrow$$

$$|\arg(z)| < \frac{\pi}{2}$$

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$$\mathscr{B}[S_0](\zeta) = \frac{1}{\zeta}\left(\frac{1}{e^{\zeta}-1} - \frac{1}{\zeta} + \frac{1}{2}\right)$$

Borel transform

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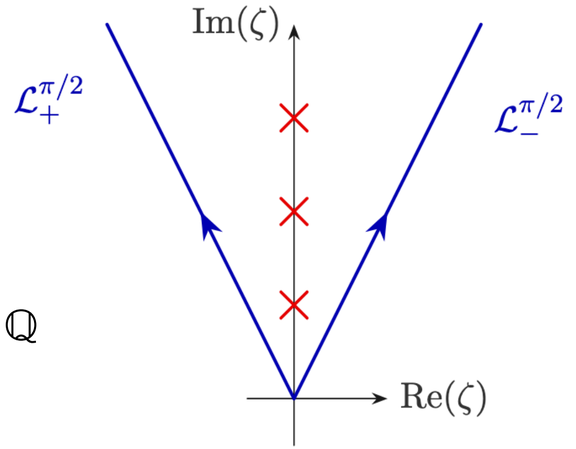
Borel resummation

$$\mathscr{L}^{\theta}[S](z) = \int_{\mathscr{C}^{\theta}} \mathrm{d}\zeta\, \mathscr{B}[S](\zeta)e^{-z\zeta}, \quad \text{where } \theta = \arg(z)$$

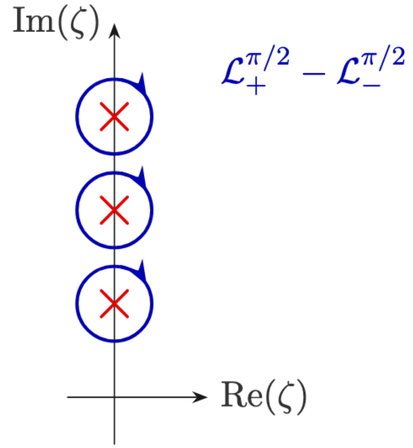
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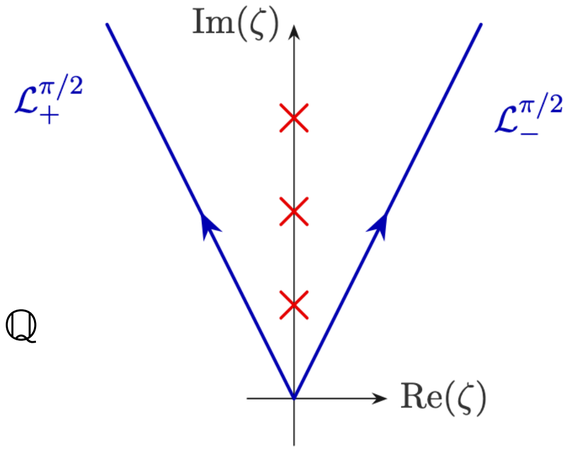
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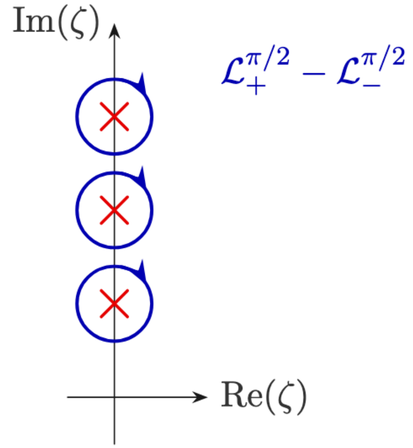
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Transseries [Nemes '22]

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$$|\arg(z)| < \pi$$

$$\mathrm{S}_k^{+}(\theta) = \begin{cases} 0 & -\pi < \theta < \frac{\pi}{2} \\ \frac{1}{2} & \theta = +\frac{\pi}{2} \\ 1 & \frac{\pi}{2} < \theta < \pi \end{cases}$$

Transseries from difference equations, I

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$z = \infty$ is an irregular singularity

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- For n-point: contiguity $\quad \Phi(\mathbf{s}+e_j)=\mathsf{A}_j(\mathbf{s})\,\Phi(\mathbf{s}) \quad j=1,\dots,\frac{n}{2}(n-3)$ $\qquad \Phi=\left(\int_C e^{\lambda W}\varphi_\rho\right)_\rho \qquad \mathsf{A}_j\in \mathrm{GL}_{(n-3)!}(\mathbb{C}(\mathbf{s}))$

  flat discrete connection

Transseries from difference equations, II

- Projection/iteration along a **rational** direction fixes the shape of the asymptotics:

we find $c_{k,\sigma} \subset B_k \in \mathbb{Q}$ [XK, Stieberger '26]

cf $\zeta(-n_1, \ldots, -n_r) \in \mathbb{Q}$

$$\zeta(-r_1, -r_2) = -\frac{1}{2}\zeta(-r_1 - r_2)$$

[Akiyama, Egami, Tanigawa '01]

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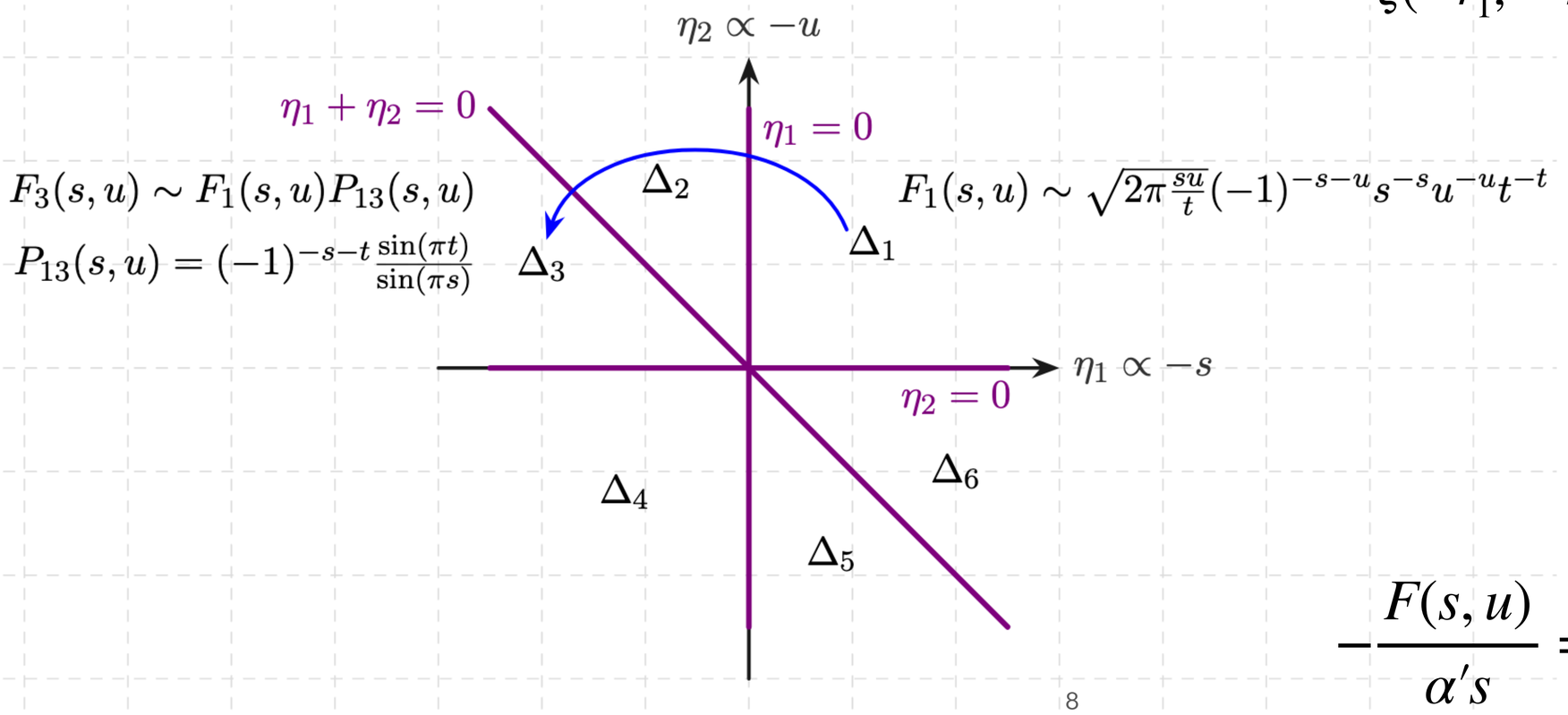
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[Akiyama, Egami, Tanigawa '01]



$F_3(s,u) \sim F_1(s,u) P_{13}(s,u)$

$P_{13}(s,u) = (-1)^{-s-t} \frac{\sin(\pi t)}{\sin(\pi s)}$

$F_1(s,u) \sim \sqrt{2\pi \frac{su}{t}} (-1)^{-s-u} s^{-s} u^{-u} t^{-t}$

BUT

blind to periodic ambiguity

already for for $n = 4$:

$$-\frac{F(s,u)}{\alpha' s} = \int_0^1 \mathrm{d}z\, z^{-\alpha' s - 1} (1-z)^{-\alpha' u}$$

$\alpha' \to 0$ and $\alpha' \to \infty$ in a single framework

- Given the Borel resummation of the $n = 4$ integral:

$$\ln F(s,u) \subset \int_0^\infty \mathrm{d}x \, \frac{K(x)}{x} \left[e^{-\alpha' s x} + e^{-\alpha' u x} - e^{-\alpha'(s+u)x} \right] \qquad \text{with} \qquad K(x) = \frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}$$

small α': Taylor expansion of $e^{-\alpha' c t}$/ large α': Bernoulli expansion of $K(t)$

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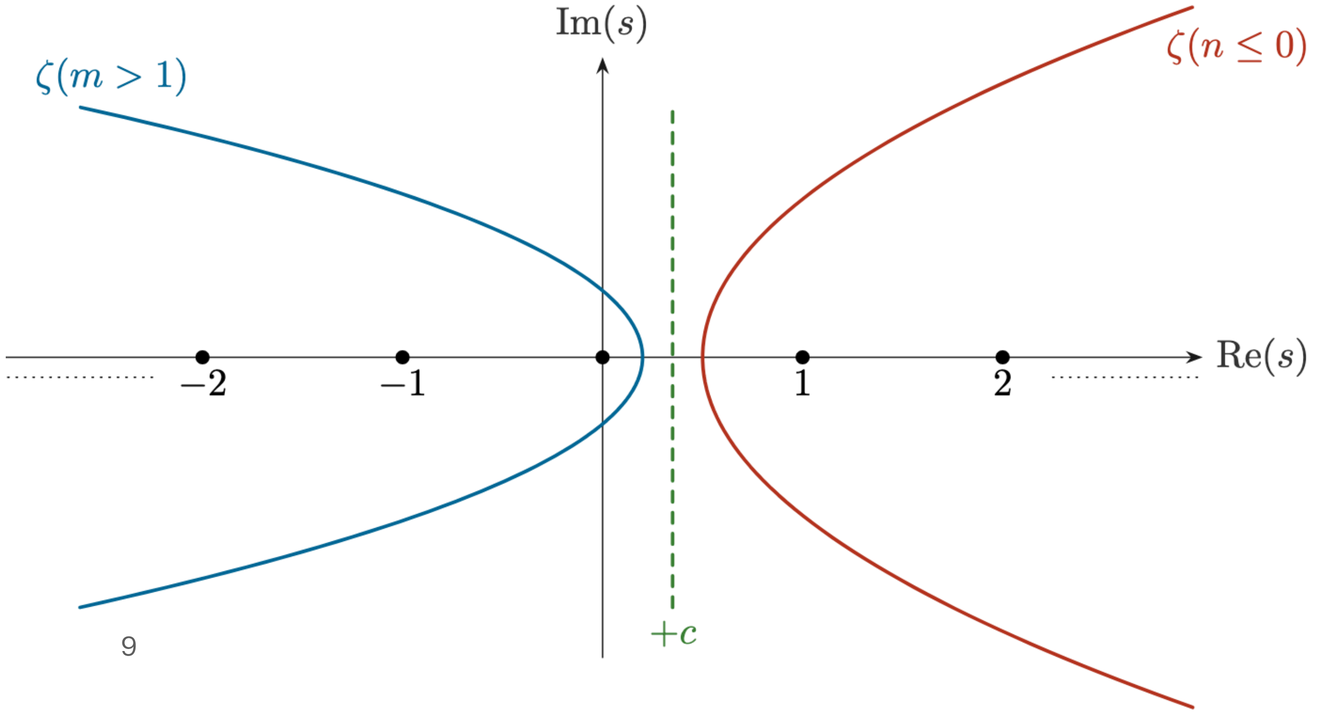
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- Single MB bridge:

$$e^{-x} = \frac{1}{2\pi i} \int_{-i\infty+c}^{+i\infty+c} \mathrm{d}s \, \Gamma(s)\, x^{-s}, \quad c > 0$$



Summary & Open Questions

- The connection problem is encoded ↗ geometrically (twisted intersection numbers of thimbles)
  ↘ algebraically through resurgence (Stokes coefficients)

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Thank you!

Backup slides

KS's framework

Forms modulo exact twisted forms on all of X

Depends homomorphically on λ

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period pairing

$$I_\Gamma(\lambda) = \langle \Gamma \otimes e^{\lambda W}, \omega \rangle$$

Laplace's method

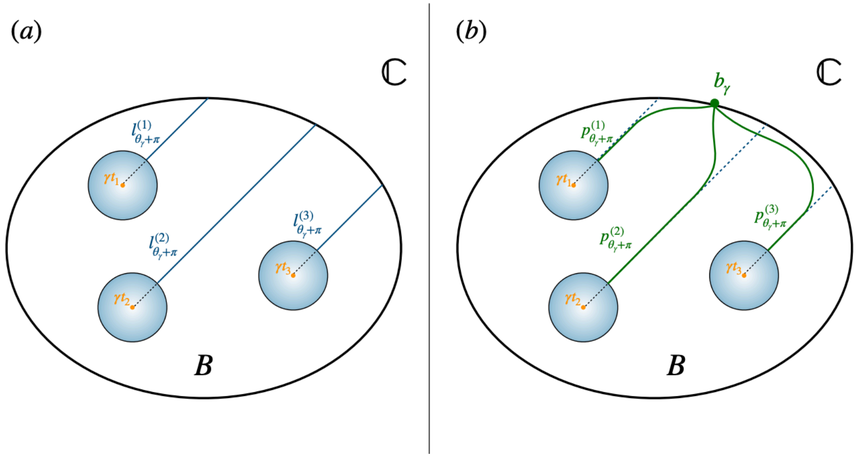
$$I_{\mathcal{J}_\sigma}(\lambda)$$

$$H^{\mathrm{Betti,\,glob},1/\lambda}_\bullet \xleftarrow{\mathrm{iso}_{\mathrm{Betti}}} H^{\mathrm{Betti,\,loc},1/\lambda}_\bullet$$

Same thing but computed in formal nbd of each critical point, with coefficients in power series

Cycles on which $e^{\lambda W}$ decays at the ends. For sv'd W this is relative homology $H_\bullet(X, \{\Re(\lambda W) \leq -c\})$ for $c \to -\infty$

For each critical value $W(s_\sigma)$, relative homology of small disc around $\lambda W(s_\sigma)$ with marked point on bdy at angle $\theta = -\arg(\lambda)$ where thimble leaves the disc.



KS's framework for the Euler Beta function

Potential: $W = -s\log x - u\log(1-x)$ $\omega = \dfrac{\mathrm{d}x}{x}$ $X = \mathbb{P}^1\backslash\{0,1,\infty\}$

Saddle: $x_* = \dfrac{s}{s+u}$ $s,u<0 \Rightarrow x_* \in (0,1)$ $(0,1)$ is a thimble

for other kinematics, $W_{n,m}(x_*) = W(x_*) - 2\pi\mathrm{i}(ns+mu)$

From KS POV:

Local Betti group: $H_1^{\text{Betti,loc},1/\lambda} \cong \mathbb{Z}$ since only one zero of $\mathrm{d}W$: generated by thimble through x_*

Lifts $\mathscr{J}_{n,m}$ on universal cover all map to same generator, $\Gamma = (0,1)\otimes e^{\lambda W}$

mult. by loading $q_0^m q_1^n = e^{-2\pi\mathrm{i}(ms+nu)}$

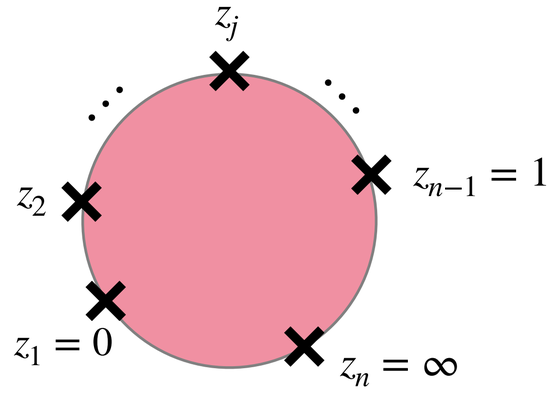
$\Rightarrow$ Global Betti $\cong \mathbb{C}$, $\text{iso}_{\text{Betti}}$ is a 1×1 matrix $\Gamma = H_1(X,\{x_*\};\mathbb{Z}) = \mathbb{Z}\gamma_0 \oplus \mathbb{Z}\gamma_1$, $Z(\gamma_i) = 2\pi\mathrm{i}\text{Res}_i\alpha$

Tree-level open string amplitudes

- n-point building blocks: [Mafra, Schlotterer, Stieberger '11]

  basis of $(n-3)!$ Euler integrals on disk with marked points

  $\mathscr{M}_{0,n}(\mathbb{R}) = \{(z_2, z_3, \ldots, z_{n-2}) \in (\mathbb{RP}^1)^{n-3} \mid z_i \neq z_j \text{ for } i \neq j\}$



$$Z_{\pi\rho}(\alpha') = \left(\prod_{j=2}^{n-2} \int_{C_\pi} \mathrm{d}z_j \right) \frac{\prod_{i<j}^{n-1} |z_{ij}|^{-\alpha' s_{ij}}}{z_{12_\rho} z_{2_\rho 3_\rho} \cdots z_{(n-3)_\rho (n-2)_\rho}}$$

$s_{ij} = (k_i + k_j)^2$

$$C_\pi = \{z_j \in \mathbb{R} \mid 0 < z_{\pi(2)} < \ldots < z_{\pi(n-2)} < 1\}$$

Low- vs high-energy limits

- $\alpha' \to 0$: field-theory limit

$$Z_{\pi\rho}(\alpha') = (-\alpha')^{3-n}\ S^{-1}_{\pi\rho} + \sum_{k=2}^{\infty} (-\alpha')^{3-n+k}\ b_k$$

momentum kernel

[Kawai, Lewellen, Tye '85] [Bern, Dixon, Perelstein, Rozowsky '99]
[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove '11]

(polynomials of weight $3+n-k$ in the s_{ij}'s) $\times$ (MZVs of weight k)

$$\zeta(n_1, n_2, \ldots, n_r) = \sum_{0<k_1<k_2<\ldots<k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \ldots k_r^{n_r}}$$

$$n_i \in \mathbb{N}^*$$

[Brown '09] [Stieberger '11]
[Schlotterer, Stieberger '13]
[Zagier '94]

- $\alpha' \to \infty$: saddle points and thimbles

$$Z_{\pi\rho}(\alpha') = \oint_{\mathcal{J}} \mathrm{d}^{n-3}\vec{z}\ g(\vec{z})\ e^{-\alpha' S}$$

thimble

Morse function

$$S(\{z_l\}; \hat{s}) = \sum_{i<j} \hat{s}_{ij}\ \ln(z_i - z_j)$$

$(n-3)!$ scattering equations:

$$\frac{\partial S}{\partial z_i} = \sum_{j\neq i} \frac{\hat{s}_{ij}}{z_i - z_j} = 0$$

???

asymptotic series:

$$Z^{(i)}_{\rho}(\alpha') = \left[\det H(\{z^{(i)}_l\})\right]^{-1/2}\ e^{-\alpha' S(\{z^{(i)}_l\})} \sum_{k\geq 0} \frac{c^{(i)}_k}{(-\alpha')^k}$$

Thimble analysis of the four-point string amplitude

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$$F(s,u) = -s\int_0^1 \mathrm{d}z\, z^{-s-1}(1-z)^{-u} \qquad \Re(s) < 0, \Re(u) < 1$$

unphysical!

unphysical: saddle $x_0 \in (0,1)$

$$F(s,u) \sim \sqrt{2\pi\frac{su}{t}}e^{-s\ln(s)-u\ln(u)-t\ln(t)}(-1)^{-s-u}$$

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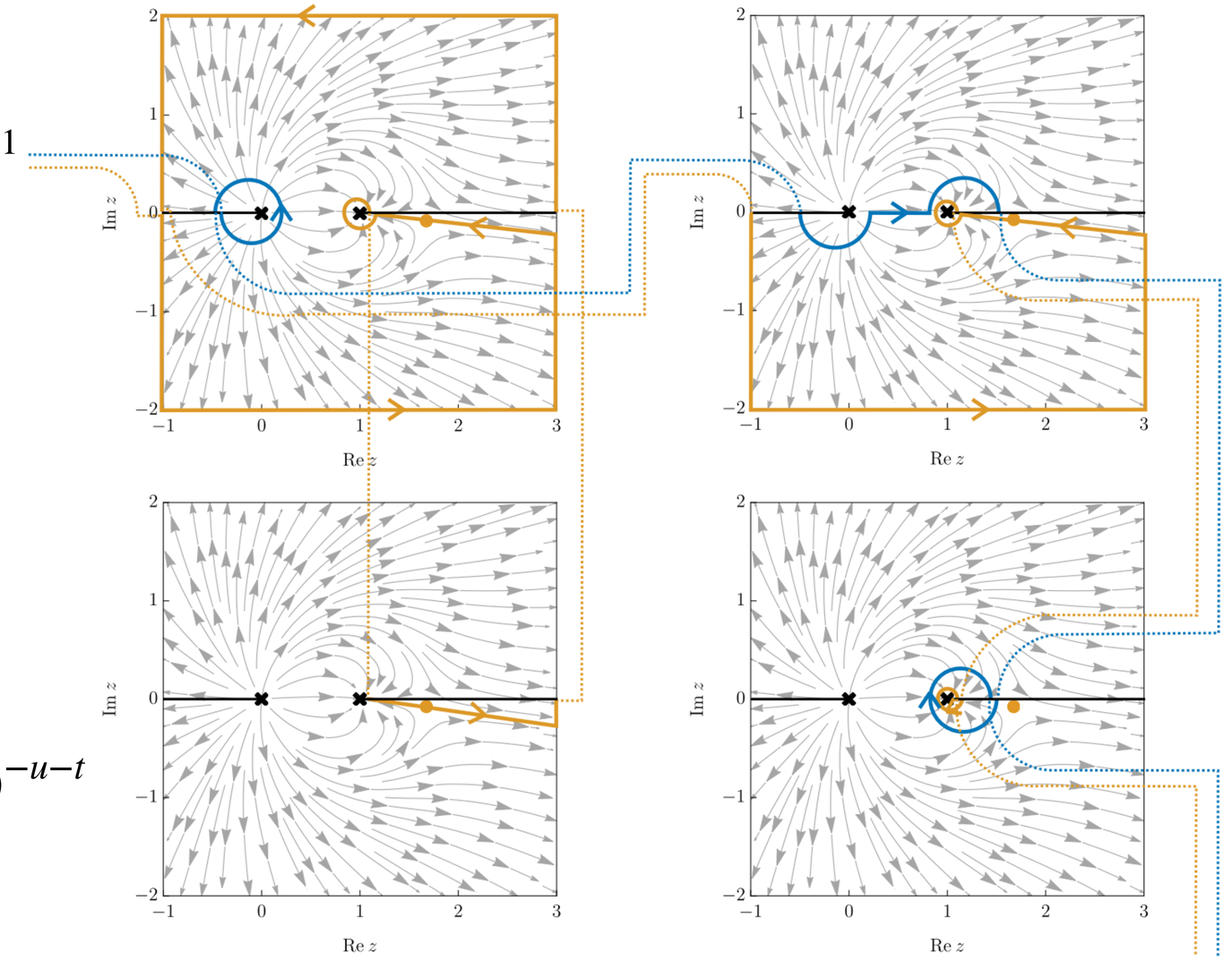
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physical: saddle $x_0 \notin (0,1)$ + complex saddles

$$F(s,u) \sim \sqrt{2\pi\frac{su}{t}\frac{\sin(\pi t)}{\sin(\pi s)}}e^{-s\ln(s)-t\ln(t)-u\ln(u)}(-1)^{-u-t}$$



[Witten '13]

figure from [Yoda '24]